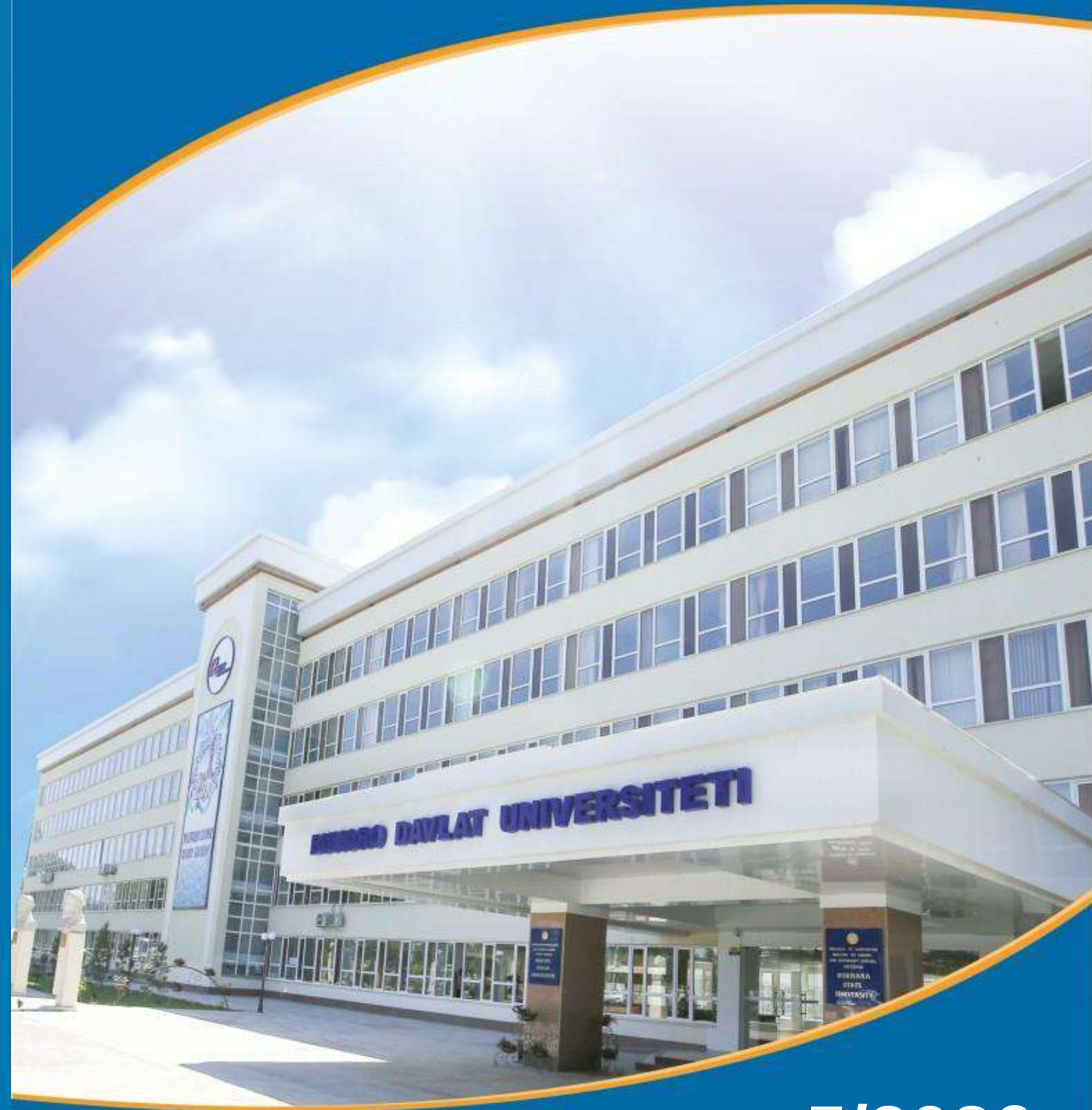


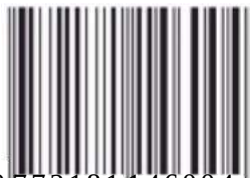
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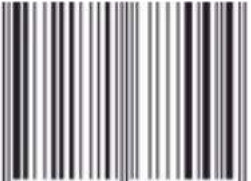
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ОЦЕНКА СКОРОСТИ СХОДИМОСТИ В ПРЕДЕЛЬНЫХ ТЕОРЕМАХ О ПЕРЕХОДНЫХ ЯВЛЕНИЯХ ДЛЯ ВЕТВЯЩИХСЯ СЛУЧАЙНЫХ ПРОЦЕССОВ НАЧИНАЮЩИХСЯ СО СЛУЧАЙНОГО ЧИСЛА ЧАСТИЦ

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Аннотация. В предлагаемой работе исследуются оценки скорости сходимости в переходных явлениях, т.е. в предельных теоремах для ветвящихся случайных процессов при $A \rightarrow 1, n \rightarrow \infty$. Получена скорость сходимости остаточного члена в предельных теоремах о переходных явлениях для ветвящихся процессов, начинающегося с со случайного числа частиц.

Ключевые слова: ветвящиеся случайные процессы, переходные явления, числовые характеристики, производящие функции, оценки скорости сходимости в переходных явлениях, ветвящиеся процессы близкие к критическим.

TASODIFIY SONDAGI ZARRACHALARDAN BOSHLANUVCHI TARMOQLANUVCHI TASODIFIY JARAYONLAR UCHUN O'TISH HOLATLARI HAQIDAGI LIMIT TEOREMALARDA YAQINLASHISH TEZLIGINI BAHOLASH

Annotatsiya. Taklif etilayotgan ishda o'tish holatlarida, ya'ni tasodifiy sondagi zarrachalardan boshlanuvchi tarmoqlanuvchi tasodifiy jarayonlar uchun limit teoremlarida yaqinlashish tezligini baholash o'rganiladi. Tarmoqlanuvchi jarayonlar uchun, tasodifiy sondagi zarrachalardan boshlanadigan, o'tish holatlari haqidagi limit teoremlarda qoldiq hadning yaqinlashish tezligi olingan.

Kalit so'zlar: tarmoqlanuvchi tasodifiy jarayonlar, o'tish holatlari, hosil qiluvchi funksiyalar, o'tish holatlarida yaqinlashish tezligini baholash, kritikga yaqin tarmoqlanuvchi jarayonlar.

ESTIMATION OF CONVERGENCE RATE IN LIMIT THEOREMS ON TRANSIENT PHENOMENA FOR BRANCHING RANDOM PROCESSES STARTING WITH A RANDOM NUMBER OF PARTICLES

Abstract. The present work investigates estimates of the rate of convergence in transitional phenomena, i.e., in limit theorems for branching stochastic processes as $A \rightarrow 1, n \rightarrow \infty$. The rate of convergence of the remainder term in the limit theorems concerning transitional phenomena for branching processes starting with a random number of particles is obtained.

Key words: branching stochastic processes, transitional phenomena, numerical characteristics, generating functions, estimates of the rate of convergence in transitional phenomena, nearly critical branching processes.

Введение. Предварительные результаты. Моделью многочисленных реальных явлений размножения и превращения частиц в естественных и общественных науках является теория ветвящихся процессов. Важную роль при изучении задач в демографии, медицине, химии, биологии, а также при генетическом кодировании и в управлении популяцией играют ветвящиеся случайные процессы.

Изучение асимптотических свойств ветвящихся случайных процессов, близких к критическому, является одним из многочисленных важных задач в современной теории вероятностей.

Для ветвящихся процессов, было доказано ряд предельных теорем с переходными явлениями. Однако, оценка скорости сходимости остаточных членов в таких предельных теоремах.

В теоремах с переходными явлениями для ветвящихся случайных процессов, начинающихся со случайного числа частиц, оценка скорости остаточных членов и предельные теоремы также осталось не раскрытой.

Из выше сказанного, следует, основной целью нашего исследования является изучение асимптотических свойств ветвящихся процессов, близких к критическому и изучение предельных

теорем в переходных явлениях для ветвящихся процессов, начинающихся со случайного числа частиц.

В монографиях [1]-[4] можно найти сведения, в которых изложение последующие развития этой теории.

Переходные явления для процессов Гальтона – Ватсона исследованы в работах ([1]-[5], [10]-[11]). Полезные и очень точные неравенства для распределений числа частиц n -го поколения доказаны в работе [2].

Оценка скорости сходимости в основной лемме критического процесса Гальтона – Ватсона впервые приведена в работе [15].

Данная формула

$$Z_0 = 1, Z_n = \sum_{j=1}^{Z_{n-1}} \xi_j^{(n)}, \quad n \geq 1 \quad (1)$$

определяет ветвящийся случайный процесс Гальтона – Ватсона, где $\xi_1, \xi_2, \dots, \xi_n, \dots$

последовательность независимых случайных величин (с. в.) с неотрицательными и целочисленными значениями и общим распределением

$$P(\xi_1 = k) = P(Z_1 = k) = p_k, \quad k = 0, 1, \dots$$

(см. [1], гл.1, §1, стр.11-13, [2], ch.1, part A, §1, 1-4 pp.).

Из определения ветвящегося процесса Гальтона – Ватсона, равенство (1), следует, что этот процесс представляет собой однородную цепь Маркова с фазовым множеством состояний $\{0, 1, 2, \dots, n, \dots\}$ и переходными вероятностями

$$p_{ij}(n) = P(Z_{n+1} = j / Z_n = i) = p_{1j}^{*i} = \sum_{j_1 + \dots + j_i = j} p_{1j_1}(n) p_{1j_2}(n) \dots p_{1j_i}(n).$$

где $i, j = 0, 1, 2, \dots$,

$$p_{1j}(n) = P(Z_n = j / Z_0 = 1) = P(Z_n = j),$$

$$p_{0j}(n) = p_{1j}^{*0}(n) = \delta_{ij} = \begin{cases} 1 & \text{при } i = j, \\ 0 & \text{при } i \neq j. \end{cases}$$

Пусть $F(x)$ - производящая функция с. в. ξ_1 т. е.

$$F(x) = Ex^{\xi_1} = \sum_{k=0}^{\infty} P(\xi_1 = k) x^k, \quad |x| \leq 1.$$

и пусть V — произвольная неотрицательная целочисленная с. в. с производящей функцией

$$G(x) = Ex^V = \sum_{k=0}^{\infty} P(V = k) x^k, \quad |x| \leq 1$$

Имеет место следующее:

Предложение. Пусть последовательность с. в.

$$V, \xi_1, \xi_2, \dots, \xi_n, \dots$$

независима в совокупности.

Тогда производящая функция случайной суммы $S_v = \sum_{j=1}^v \xi_j$ удовлетворяет равенству

$$Ex^{S_v} = \sum_{k=0}^{\infty} P(S_v = k) x^k = G(F(x)), \quad |x| \leq 1.$$

Пользуясь приведённым предложением, можно установить рекуррентные отношения для производящих функций

$$F_n(x) = \sum_{k=0}^{\infty} P(Z_n = k) x^k = \sum_{k=0}^{\infty} P_n(k) x^k, \quad n = 1, 2, \dots$$

а именно, имеют место равенства

$$F_n(x) = F_{n-1}(F(x)) = F(F_{n-1}(x)), \quad n \geq 1. \quad (2)$$

Формулы (2) легко могут быть доказаны методом математической индукции с учетом предложения и равенства (1).

Обычно, в теории ветвящихся процессов используют следующие обозначения:

$$A = EZ_1 = F'(1), \quad B = EZ_1(Z_1 - 1) = F''(1), \quad C = EZ_1(Z_1 - 1)(Z_1 - 2) = F'''(1).$$

Процесс Гальтона – Ватсона называется докритическим, критическим и надкритическим, если $0 < A < 1$, $A = 1$, $A > 1$, соответственно.

В критическом случае $A = 1$, предельным распределением для Z_n является конкретное показательное распределение, а в некритических случаях $A \neq 1$ эти предельные распределения определяются сложными функциональными уравнениями, не имеющими явных решений.

Явления, возникающие при $n \rightarrow \infty$, $A \rightarrow 1$, т.е. асимптотические свойства процессов, близких к критическим, будем называть переходными явлениями. Изучение переходных явлений основывается на асимптотической формуле для $1 - F_n(x)$ при $n \rightarrow \infty$, $A \rightarrow 1$. Вид этой формулы можно определить на конкретном примере ветвящихся процессов Гальтона – Ватсона, порождённых дробно-линейными производящими функциями вида

$$F(x) = \frac{ax + b}{cx + d}.$$

Как замечено в монографии [1] (гл. III, §4, стр. 108), для производящей функции $F(x)$ из формулы (2) итерации $F_n(x) = F(F_{n-1}(x))$ находятся в виде явного выражения при $n \rightarrow \infty$, $A \rightarrow 1$

$$r_n(x) = \begin{cases} \frac{A^n(1-x)}{1 + \frac{B}{2} \cdot \frac{A^n - 1}{A - 1}(1-x)}, & \text{при } A \neq 1, \\ \frac{1-x}{1 + \frac{Bn}{2}(1-x)}, & \text{при } A = 1, \end{cases} \quad (3)$$

Следуя Б. А. Севастьянову (Теор. вер. и ее прим. т.4, №2 (1959), 121-135), определим класс вероятностных производящих функций

$$K(B_0, C_0) = \{F(\cdot), F''(1) = B \geq B_0 > 0, F'''(1) = C \leq C_0 < \infty\}.$$

Первым результатом о переходных явлениях для процессов Гальтона – Ватсона является следующая

Теорема 1. Выполняется равенство

$$1 - F_n(x) = r_n(x)(1 + \eta_n(x))$$

где $r_n(x)$ определяется формулой (1.5), а $\eta_n(x) \rightarrow 0$ при $n \rightarrow \infty$, $A \rightarrow 1$ равномерно по всем $F(x) \in K(B_0, C_0)$ и $|x| \leq 1$.

Эта теорема, доказанная Б. А. Севастьяновым, содержится в монографии [1] (гл. III, §4, стр. 106).

В работе [31] доказано оценка для остаточного члена теоремы 1. Исследуем скорость сходимости к нулю бесконечно малой величины $\eta_n(A, x)$ равномерно по классу $K(B_0, C_0)$.

Соответствующие результаты приведем в следующий теореме.

Теорема 2. При $n \rightarrow \infty, A \rightarrow 1$ и $|1-x| \geq r > 0$ бесконечная малая функция $\eta_n(A, x)$ в теореме 1 имеет следующие оценки:

1°. Если $\min\left(n, \frac{1}{|A-1|}\right) = n$, то

$$\sup_{|x-1| \geq r > 0} |\eta_n(A, x)| = O\left(\frac{\ln g(n, A)}{g(n, A)}\right); \quad (4)$$

2°. Если $\min\left(n, \frac{1}{|A-1|}\right) = \frac{1}{|A-1|}$, то

$$\sup_{|x-1| \geq r > 0} |\eta_n(A, x)| = O(|A-1|) + O\left(\frac{\ln g(n, A)}{g(n, A)}\right); \quad (5)$$

равномерно по классу производящих функций $F(x) \in K(B_0, C_0)$.

Доказательство теоремы 2 основывается на нижеследующих вспомогательных утверждениях.

Лемма 1. Функция $g(n, A)$ возрастает по каждому аргументу при фиксированном втором аргументе. При всех $n \geq 2$ и $A > 0$ имеет место неравенство

$$g(n, A) \geq \frac{1}{2} \min\left(n, \frac{1}{|1-A|}\right)$$

Эта лемма приведена в монографии [1] (гл. III, §4, стр. 104).

Основные результаты. Теперь рассмотрим ветвящийся случайный процесс с дискретным временем начинающихся со случайного числа ζ частиц. Предполагается, что случайная величина ζ – положительная и целочисленная с математическим ожиданием $E\zeta = m$ и дисперсией $D\zeta = \sigma^2 > 0$.

Пусть $Z_{n\zeta}$ – число частиц к моменту времени n . Как легко следует из определения рассматриваемого процесса, случайная величина (с. в.) $Z_{n\zeta}$ представимо в виде суммы независимых, одинаково распределённых случайных величин со случайного числа слагаемых:

$$Z_{n\zeta} = Z_1^{(n)} + Z_2^{(n)} + \dots + Z_\zeta^{(n)}, \quad (Z_i^{(0)} = 1)$$

где $Z_i^{(n)}$ – число частиц, произведённое i -ой частицей за n поколений. (i – номер частицы), причём с. в. $Z_i^{(n)}$ – одинаково распределены с Z_n – число частиц к моменту времени n в процесс Гальтона – Ватсона начинающегося с одной частицы.

Введем теперь производящие функции рассматриваемых здесь случайных величин.

$$\Lambda(s) = \sum_{k=1}^{\infty} P(\zeta = k) s^k = \sum_{k=1}^{\infty} w_k s^k, \quad F_n(s) = \sum_{k=0}^{\infty} P(Z_n = k) s^k = \sum_{k=0}^{\infty} P_{nk} s^k, \quad |s| \leq 1, \quad F(s) = F_1(s).$$

Как легко видеть, что $\sum_{k=0}^{\infty} P(Z_{n\zeta} = k) s^k = \Lambda(F_n(s))$ и пусть

$$\bar{Q}_n(s) = 1 - \Lambda(F_n(s)), \quad \bar{Q}_n = 1 - \Lambda(F_n(0)).$$

Для случайной величины $Z_{n\zeta}$ легко вычислить математического ожидание и дисперсии, которые имеют в следующем виде:

$$EZ_{n\zeta} = mA^n, \quad EZ_{n\zeta}^2 = mDZ_n + A^{2n}E\zeta^2$$

$$DZ_{n\zeta} = \sigma_n^2 = \begin{cases} m \frac{A^n(A^n-1)(B+A-A^2)}{A^2-A} + A^{2n}\sigma^2, & \text{если } A > 1, \\ mnB + \sigma^2, & \text{если } A = 1, \\ m \frac{A^n(1-A^n)(B+A-A^2)}{A-A^2} + A^{2n}\sigma^2, & \text{если } A < 1. \end{cases}$$

где $A = F'(1)$, $B = F''(1)$, $C = F'''(1)$.

Аналогично, условные математическое ожидание с. в. $Z_{n\zeta}$ имеет вид:

$$E(Z_{n\zeta} / Z_{n\zeta} > 0) = \frac{EZ_{n\zeta}}{P(Z_{n\zeta} > 0)} = \frac{mA^n}{1 - \Lambda(F_n(0))} = \frac{mA^n}{\bar{Q}_n}.$$

Положим $S_{n\zeta}(x) = P\left\{\frac{Z_{n\zeta}}{E(Z_{n\zeta} / Z_{n\zeta} > 0)} < x / Z_{n\zeta} > 0\right\}$ и показательной распределение

$$S(x) = \begin{cases} 1 - e^{-\frac{x}{m}}, & \text{если } x > 0, \\ 0, & \text{если } x \leq 0. \end{cases}$$

Теорема 3. Если $A=1$, $0 < B < \infty$, $\sigma^2 < \infty$, то при $n \rightarrow \infty$

$$S_n(x) \rightarrow S(x).$$

В переходных явлениях в теории ветвящихся случайных процессов, следуя Б. А. Севастьянова [1], вводят класс производящих функций (класс процессов) $K(B_0, C_0)$, удовлетворяющих условиям $K(B_0, C_0) = \{F : F'(1) = B \geq B_0 > 0, F''(1) = C \leq C_0 < \infty\}$.

Пользуясь леммой 1 и теорема 2 докажем следующую теорему:

Теорема 4. Пусть $n \rightarrow \infty$, $A \rightarrow 1$ и $|s-1| \geq \varepsilon > 0$.

Тогда выполняется равенство

$$\bar{Q}_n(s) = \bar{r}_n(s)(1 + \eta_{n\zeta}(s))$$

$$\text{где } \bar{r}_n(s) = \frac{mA^n(1-s)}{1 + \frac{B}{2}g(n, A)(1-s)}$$

и

$$\sup_{|s-1| \geq \varepsilon > 0} |\eta_{n\zeta}(s)| = O\left(\max\left(\frac{\ln g(n, A)}{g(n, A)}, |A-1|\right)\right)$$

равномерно по всем производящим функциям $F(s) \in K(B_0, C_0)$ и $|s| \leq 1$.

Доказательство теоремы 4. Так как с. в. ζ имеет математическое ожидание m и дисперсией $\sigma^2 > 0$ то используя разложение Тейлора имеем

$$\Lambda(s) = 1 + m(s-1) + O((s-1)^2)$$

Отсюда, заменяя s на $F_n(s)$ получаем

$$1 - \Lambda(F_n(s)) = m(1 - F_n(s)) + O((1 - F_n(s))^2) \quad (6)$$

или $\bar{Q}_n(s) = mQ_n(s) + O(Q_n^2(s))$.

Отсюда и из теоремы 1 – 3 легко следует утверждений теоремы 4.

Теорема 4 доказана.

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PANJARADAGI IKKI ZARRACHALI SISTEMAGA MOS SHRYODINGER OPERATORI
XOS QIYMATLARINI ASIMPTOTIK HOLATLARI

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Annotatsiya. Ushbu maqolada ikki o'lchamli $\mathbb{Z}^2$ panjarada xarakatlanuvchi va qisqa masofada ta'sirlashuvchi ikkita bir xil zarrachali (fermionli) sistemaga mos Shredinger operatorlarining muhim spektri tubida yotuvchi kamida bitta xos qiymati mavjud bo'lishi ko'rsatilgan va kvaziimpulsining kichik qo'zg'alishlarda xos qiymati uchun asimptotik yoyilmalar olingan va ularga mos xos funksiyalarni qurilgan.

Kalit so'zlar: Hamiltonian, bog'langan holat, Shryodinger operatori, to'la kvaziimpuls, xos qiymat, qo'zg'alish nazariyasi, Birman-Shvinger prinsipi.

ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF THE SCHRÖDINGER
OPERATOR CORRESPONDING TO A TWO-PARTICLE SYSTEM ON A LATTICE.

Abstract. In this paper, eigenvalues outside the essential spectrum and the explicit form of the corresponding eigenfunctions of the two-particle Schrödinger operator are found. This operator corresponds to a system of two identical particles (fermions) moving on a two-dimensional lattice $\mathbb{Z}^2$ with short-range and weak long-range interaction potentials. Additionally, asymptotic expansions of the eigenvalues for small perturbations of the quasi-momentum are obtained.

Key words: Hamiltonian, bound state, Schrödinger operator, total quasimomentum, eigenvalue, theory perurbation, Birman-Schwinger principle.

АСИМПТОТИЧЕСКИЕ ПОВЕДЕНИЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ ОПЕРАТОРА
ШРЕДИНГЕРА СООТВЕТСТВУЮЩЕЙ СИСТЕМЕ ДВУХ ЧАСТИЦ НА РЕШЕТКЕ

Аннотация. В данной статье найдены собственные значения вне существенного спектра и явный вид соответствующих собственных функции двухчастичного оператора Шредингера, соответствующего системе двух одинаковых частиц (фермионов) короткодействующим и слабо дальнодействующим потенциалом, движущихся на двумерной решетке $\mathbb{Z}^2$ и найдены асимптотические разложения для собственных значений при малых возмущениях квазиимпульса

Ключевые слова: Гамильтониан, связанное состояние, оператор Шредингера, полный квазиимпульс, собственное значение, теория возмущений, принцип Бирмана-Швингера.

Kirish. Faraz qilaylik, ikki o'lchamli $T^2 \equiv (-\pi, \pi]^2$ tor (Brillyuen zonasi) berilgan bo'lib, bunda $T^2 \equiv (-\pi, \pi]^2 \subset P^2$ to'plam elementlarini qo'shish va haqiqiy songa ko'paytirish amallari har doim P^2 da $(2\pi\mathbb{Z}^2)$ modul bo'yicha hisoblanadi.

Kichik parametr qiymatlarida ikki zarrali klaster operatorlarining bog'langan holatlari paydo bo'lishining tabiati ilk bor R. A. Minlos va Sh. Mamatov tomonidan [1] da o'rganilgan, keyinchalik esa yanada umumiy holatda R. A. Minlos va A. I. Mogilner tomonidan [2] da o'rganilgan. Ikki zarrachali sistemaga mos $\mathbb{Z}^d$ panjaradagi H Hamiltonianning bog'langan holatlarini o'rganish $H(\mathbf{k})$ $\mathbf{k} \in T^d = (-\pi, \pi]^d$ Shredinger operatorlari oilasining xos qiymatlarini tekshirishga keltiriladi. Bunda $H(\mathbf{k})$, operatorining xos funksiyalari H Hmiltonianning bog'langan holatlari sifatida, xos qiymatlari esa bog'langan holat energiyalari sifatida talqin qilinadi. Bir o'lchamli panjaradagi ikki fermionli sistemaga mos H Hamiltonianning bog'langan holatlari [4] da o'rganilgan, bir o'lchamli panjaradagi ikki zarrachali Shredinger operatorining xos qiymatlarining qo'zg'alishlari esa [5] ishda tadqiq etilgan. J.I. Abdullayevning [7] ishida H operatorning xos qiymatining karraliligi o'rganilgan. J.I. Abdullayev, I.A. Ikromov va S.N. Lakayevning [8] ishida operatorning xos qiymatlari va rezonanslari tadqiq etilgan.

Mazkur ishda ikki o'lchamli $\check{y}^2$ panjarada ikki fermionli sistemaga mos H (ga ko'ra (2.1)) Hamiltonianining bog'langan holatlari qaralib, ya'ni ushbu $\hat{H}$ Hamiltonianiga mos keluvchi $H(\mathbf{k})$, $\mathbf{k} = (k_1, k_2) \in T^2$, Shredinger operatorlari oilasining diskret spektri (qarang (2.3)) o'rganiladi. (2.3) va (2.4) dan $H(k_1, k_2) = H(-k_1, k_2) = H(k_1, -k_2)$ kelib chiqadi, shuning uchun $k_1, k_2 \in [0, \pi]$ deb olish mumkin.

Agar $\mathbf{k} = (\pi, \pi)$ bo'lsa, u holda $H(\pi, \pi)$ operator $4 - \hat{v}(\mathbf{n}), \mathbf{n} \in \check{y}^2$ ko'rinishdagi cheksiz ko'p xos qiymatlarga ega bo'lib, muhim spektr bitta $\{4\}$ nuqtadan iborat bo'ladi.

2. Kordinata tasvirdan impuls tasvirga o'tish

Z^2 panjarafagi ikki fermionli sistemaga mos H_0 erkin hamiltonian chegarlangan o'z-o'ziga qo'shma operator bo'lib, $l_2^{as}(\check{y}^2 \times \check{y}^2) := \{f \in l_2(\check{y}^2 \times \check{y}^2) : f(\mathbf{x}, \mathbf{y}) = -f(\mathbf{y}, \mathbf{x})\}$ Hilbert fazosida quyidagicha tasvirlanadi:

$$H_0 = -\frac{1}{2m}\Delta_1 - \frac{1}{2m}\Delta_2.$$

Bunda m fermion og'irligi bo'lib, uni bir birlik sifatida hisoblaymiz. $\Delta_1 = \Delta \otimes I$ va $\Delta_2 = I \otimes \Delta$ panjaraviy Laplas operatori bo'lib, Δ – esa

$$(\Delta \hat{\psi})(\mathbf{x}) = \sum_{j=1}^2 [\hat{\psi}(\mathbf{x} + \mathbf{e}_j) + \hat{\psi}(\mathbf{x} - \mathbf{e}_j) - 2\hat{\psi}(\mathbf{x})], \quad \mathbf{x} \in \check{y}^2, \quad \hat{\psi} \in l_2(\check{y}^2),$$

zarrachalarni bir tugundan boshqa tugunga o'tishini ifodalovchi ayirma operatoridir. $\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)$ lar Z^2 dagi birlik operatorlar. H Hamiltonian $l_2^{as}(Z^2 \times Z^2)$ Hilbert fazosida erkin Hamiltonian va ikki zarrachani tasirlashish potentsiali V_2 bilan ayrimasini ifodalaydi.

$$H = H_0 - V_2, \quad (2.1)$$

bu holda

$$(\hat{V}_2 \hat{\psi})(\mathbf{x}, \mathbf{y}) = \hat{v}(\mathbf{x} - \mathbf{y}) \hat{\psi}(\mathbf{x}, \mathbf{y}), \quad \hat{\psi} \in l_2^{as}(\check{y}^2 \times \check{y}^2).$$

$\hat{v}$ funksiya uchun quyidagi shartlarga ega bo'lsin,

$$\hat{v} \in l_2(\check{y}^2) \quad \text{va} \quad \hat{v}(\mathbf{x}) = \hat{v}(-\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \check{y}^2. \quad (2.2)$$

Bunda H Hamiltonian $l_2^{as}(\check{y}^2 \times \check{y}^2)$ fazoda chegarlangan o'z-o'ziga qo'shma operator bo'aladi.

Fur'e almashtirishlardan ikki fermionli H Hamiltonianni spektral xossalarni o'rganish ([4], [6]) masalasi $L_2^o(T^2) = \{f \in L_2(T^2) : f(-\mathbf{q}) = -f(\mathbf{q})\}$ fazodagi $H(\mathbf{k}) = H_0(\mathbf{k}) - V, \mathbf{k} \in T^2$ Shredinger operatorlar oilasini o'rganish masalasiga keladi va $L_2^o(T^2) = \{f \in L_2(T^2) : f(-\mathbf{q}) = -f(\mathbf{q})\}$ fazoda quyidagi formula bilan aniqlanadi:

$$(H(\mathbf{k})f)(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\mathbf{q})f(\mathbf{q}) - \frac{1}{2\pi} \int_{T^2} v(\mathbf{q} - \mathbf{s})f(\mathbf{s})d\mathbf{s}. \quad (2.3)$$

Bunda qo'zg'almas $H_0(\mathbf{k})$ operator $L_2^o(T^2)$ da

$$\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon\left(\frac{\mathbf{k}}{2} + \mathbf{q}\right) + \varepsilon\left(\frac{\mathbf{k}}{2} - \mathbf{q}\right) = 4 - 2\cos\frac{k_1}{2}\cos q_1 - 2\cos\frac{k_2}{2}\cos q_2 \quad (2.4)$$

funksiyaga ko'paytirish operatori.

Muhim spektr hakidagi Veyl teoremasiga asosan $H(\mathbf{k})$ operatorning muhim spektri qo'zg'almas $H_0(\mathbf{k}), \mathbf{k} \in T^2$ operatorning spektri bilan ustma-ust tushadi, ya'ni

$$\sigma_{ess}(H(\mathbf{k})) = \sigma(H_0(\mathbf{k})) = [m(\mathbf{k}), M(\mathbf{k})],$$

bunda

$$m(\mathbf{k}) = \min_{\mathbf{q} \in T^2} \varepsilon_{\mathbf{k}}(\mathbf{q}) = \sum_{i=1}^2 (2 - 2 \cos \frac{k_i}{2}), \quad M(\mathbf{k}) = \max_{\mathbf{q} \in T^2} \varepsilon_{\mathbf{k}}(\mathbf{q}) = \sum_{i=1}^2 (2 + 2 \cos \frac{k_i}{2}).$$

3. $H(\mathbf{k})$ operator uchun invariant qism fazolar.

$H(\mathbf{k})$ operatori uchun v potensialini qo'yidagi ko'rinishda qaraymiz:

$$\hat{v}(\mathbf{x}) = \begin{cases} \bar{v}(0), & \text{agar } \mathbf{x} = 0 \\ \bar{v}(1), & \text{agar } |\mathbf{x}| = 1 \\ \bar{v}(2), & \text{agar } |\mathbf{x}| = 2 \\ 0, & \text{agar } |\mathbf{x}| \geq 3. \end{cases} \quad (3.1)$$

Bu yerda $\bar{v}(0) > \bar{v}(1) > \bar{v}(2) > 0$ ba $\mathbf{x} \in \mathbb{Z}^2$, $|\mathbf{x}| = |x_1| + |x_2|$.

V – integral operatori $L_2^0(T^2)$ fazoda qo'yidagicha bo'ladi

$$\begin{aligned} \frac{1}{2\pi} v(\mathbf{p} - \mathbf{q}) = & \frac{1}{4\pi^2} [2\bar{v}(1) \sin p_1 \sin q_1 + 2\bar{v}(1) \sin p_2 \sin q_2 + 2\bar{v}(2) \sin 2p_1 \sin 2q_1 + \\ & 2\bar{v}(2) \sin 2p_2 \sin 2q_2 + 4\bar{v}(2) \cos p_1 \cos q_1 \sin p_2 \sin q_2 + \\ & + 4\bar{v}(2) \cos p_2 \cos q_2 \sin p_1 \sin q_1]. \end{aligned} \quad (3.2)$$

Agar $\hat{v}$ potensial (3.1) ko'rinishda aniqlangan va $k_1 = k_2 = \pi$ bo'lsa, u holda $H(\pi, \pi) = 4I - V$ operatorning spektri $4 - \bar{v}(1)$, $4 - \bar{v}(2)$ va 4 xos qiymatlardan iborat bo'ladi.

$z_1(\pi, \pi) = 4 - \bar{v}(1)$ soni ikki karrali xos qiymat bo'lib, unga mos normalangan xos funksiyalar quyidagi ko'rinishda bo'ladi

$$\phi_{(1,0)}^{+-}(\mathbf{p}) = \phi_1^-(p_1) \phi_0^+(p_2) = \frac{1}{\sqrt{2}\pi} \sin p_1, \quad \phi_{(0,1)}^{+-}(\mathbf{p}) = \phi_0^+(p_1) \phi_1^-(p_2) = \frac{1}{\sqrt{2}\pi} \sin p_2. \quad (3.3)$$

$z_2(\pi, \pi) = 4 - \bar{v}(2)$ soni to'rt karrali xos qiymat bo'lib, unga mos normalangan xos funksiyalar quyidagi ko'rinishda bo'ladi

$$\begin{aligned} \phi_{(2,0)}^{+-}(\mathbf{p}) &= \phi_2^-(p_1) \phi_0^+(p_2), & \phi_{(0,2)}^{+-}(\mathbf{p}) &= \phi_0^+(p_1) \phi_2^-(p_2), \\ \phi_{(1,1)}^{+-}(\mathbf{p}) &= \phi_1^+(p_1) \phi_1^-(p_2), & \phi_{(1,1)}^{-+}(\mathbf{p}) &= \phi_1^-(p_1) \phi_1^+(p_2). \end{aligned} \quad (3.4)$$

$z_\infty(\pi, \pi) = 4$ soni cheksiz karrali xos qiymat bo'lib, unga mos xos funksiyalar quyidagi ko'rinishda bo'ladi

$$\phi_{(n,m)}^{+-}(\mathbf{p}) = \phi_n^+(p_1) \phi_m^-(p_2) \quad \phi_{(n,m)}^{-+}(\mathbf{p}) = \phi_n^-(p_1) \phi_m^+(p_2) \quad n+m \geq 3.$$

$L_2^+(T) = \{f \in L_2(T) : f(p) = f(-p)\}$ va $L_2^-(T) = \{f \in L_2(T) : f(-p) = -f(p)\}$ lar orqali mos ravishda $L_2(T)$ fazoning juft va toq qism fazolarini belgilaymiz, natijada quyidagi yoyilma o'rinni:

$$L_2^0(T^2) = L_2^{++}(T^2) \oplus L_2^{+-}(T^2), \quad (3.5)$$

bunda

$$L_2^{+-}(T^2) = L_2^-(T) \otimes L_2^+(T), \quad L_2^{++}(T^2) = L_2^+(T) \otimes L_2^+(T).$$

3.1 Lemma. $L_2^{++}(T^2)$ va $L_2^{+-}(T^2)$ qism fazolar $H(k_1, k_2)$ operatorga nisbatan invariant qism fazolar tashkil qiladi.

Isbot. Dastlab $L_2^{+-}(T^2)$ qism fazoni $H_0(k_1, k_2)$ ga keyin V ga nisbatan invariant qism fazo tashkil qilishini ko'rsatamiz. (2.4) ga ko'ra $\varepsilon_{\mathbf{k}}$ funksiya

$$L_2^{++}(T^2) = \{f \in L_2(T^2) : f(p_1, p_2) = f(-p_1, p_2) = f(-p_1, -p_2)\}.$$

fazoga tegishli bo'ladi. $f \in L_2^{+-}(T^2)$ ga ko'ra $\varepsilon_{\mathbf{k}} f \in L_2^{+-}(T^2)$ bo'ladi. Demak $L_2^{+-}(T^2)$ qism fazo $H_0(k_1, k_2)$ operatorga nisbatan invariant qism fazo tashkil qiladi.

$f \in L_2^{+}(T^2)$ da (3.2) ga ko'ra

$$(Vf)(p_1, p_2) = \frac{1}{2\pi} \int_{T^2} v(p_1 - s_1, p_2 - s_2) f(s_1, s_2) ds_1 ds_2$$

operator $L_2^{+}(T^2)$ fazoga tegishli bo'ladi. Bu esa $L_2^{+}(T^2)$ fazo $H(k_1, k_2) = H_0(k_1, k_2) - V$ operatorga nisbatan invariant qism fazo ekanligi kelib chiqadi. Bundan $H(k_1, k_2)$ operator $L_2^o(T^2)$ fazoda o'z o'ziga qo'shma ekanligidan fazoni ortogonal to'ldiruvchisi $L_2^{+-}(T^2) = (L_2^{+}(T^2))^{\perp}$ ((3.5) ga ko'ra) ham $H(k_1, k_2)$ operatorga nisbatan invariant qism fazo tashkil qiladi.

Yuqoridagi lemmadan foydalanib, $H(\mathbf{k})$ operatorning $L_2^{+-}(T^2)$ va $L_2^{+}(T^2)$ qism fazolardagi ko'rinishlarini $H^{+-}(\mathbf{k})$ va $H^{++}(\mathbf{k})$ bilan belgilab, $V^{+-} = V|_{L_2^{+-}(T^2)}$ va $V^{++} = V|_{L_2^{+}(T^2)}$ ning

$f \in L_2^{+-}(T^2)$ va $g \in L_2^{+}(T^2)$ elementlarga ta'sirini quyidagi ko'rinishda topamiz:

$$(V^{+-}f)(\mathbf{p}) = \frac{1}{2\pi^2} \int_{T^2} [\bar{v}(1) \sin p_2 \sin q_2 + \bar{v}(2) \sin 2p_2 \sin 2q_2 + 2\bar{v}(2) \cos p_1 \cos q_1 \sin p_2 \sin q_2] f(\mathbf{q}) d\mathbf{q},$$

$$(V^{++}g)(\mathbf{p}) = \frac{1}{2\pi^2} \int_{T^2} [\bar{v}(1) \sin p_1 \sin q_1 + \bar{v}(2) \sin 2p_1 \sin 2q_1 + 2\bar{v}(2) \cos p_2 \cos q_2 \sin p_1 \sin q_1] g(\mathbf{q}) d\mathbf{q}.$$

Ma'lumki, $\{\varphi_n^-(q) = \frac{1}{\sqrt{\pi}} \sin nq\}_{n \in \mathbb{N}}$ va $\{\phi_0^+(q) = \frac{1}{\sqrt{2\pi}}, \phi_n^+(q) = \frac{1}{\sqrt{\pi}} \cos nq\}_{n \in \mathbb{I}}$ vektorlar

mos ravishda $L_2^-(T)$, $L_2^+(T)$ fazolardagi ortonormal bazislar tashkil qiladi. Ixtiyoriy $n \in \mathbb{N}$ uchun $L^-(n)$ va $L^+(n-1)$ lar bilan ϕ_n^- va ϕ_{n-1}^+ vektorlarga tortilgan qism fazolarni belgilaymiz.

Quyida, $L_2^-(T)$ va $L_2^+(T)$ fazolar

$$L_2^-(T) = \bigoplus_{n=1}^{\infty} L^-(n), \quad L_2^+(T) = \bigoplus_{n=0}^{\infty} L^+(n). \quad (3.6)$$

to'g'ri yig'indilarga ajraladi. (3.6) dagi yoyilma natijasida

$$L_2^{+-}(T^2) = L_2^-(T) \otimes L_2^+(T) = \bigoplus_{n=0}^{\infty} \{L_2^-(T) \otimes L^+(n)\} = \bigoplus_{n=0}^{\infty} \mathfrak{R}_n^{+-},$$

$$L_2^{++}(T^2) = L_2^+(T) \otimes L_2^-(T) = \bigoplus_{n=1}^{\infty} \{L_2^+(T) \otimes L^-(n)\} = \bigoplus_{n=1}^{\infty} \mathfrak{R}_n^{++},$$

yoyilmalar hosil bo'ladi. Bunda $\mathfrak{R}_n^{+-} := L_2^-(T) \otimes L^+(n)$, $\mathfrak{R}_n^{++} := L_2^+(T) \otimes L^-(n)$.

3.2 Lemma. Ixtiyoriy $n \in \mathbb{N}$ uchun $\mathfrak{R}_n^{+-}$ va $\mathfrak{R}_{n-1}^{++}$ lar $H(k_1, \pi)$ operatorga nisbatan invariant qism fazo tashkil qiladi.

Isbot. Faraz qilaylik, $(f\varphi_n^-)(p_1, p_2) := f(p_1)\varphi_n^-(p_2)$, $f \in L_2^+(T)$, $\varphi_n^- \in L^-(n)$, $\mathfrak{R}_n^{+-}$ dagi ixtiyoriy element bo'lsin. $H^{+-}(k_1, \pi) = H_0(k_1, \pi) - V^{+-}$ operatorni $f\varphi_n^-$ ga ta'sirini qaraymiz,

$$(H_0(k_1, \pi) f\varphi_n^-)(p_1, p_2) = [(4 - 2 \cos \frac{k_1}{2} \cos p_1) f(p_1)] \varphi_n^-(p_2).$$

$$(V^{+-} f\varphi_n^-)(p_1, p_2) = \frac{1}{2\pi^2} \int_{T^2} [\bar{v}(1) \sin p_2 \sin q_2 + \bar{v}(2) \sin 2p_2 \sin 2q_2 + 2\bar{v}(2) \cos p_1 \cos q_1 \sin p_2 \sin q_2] f(q_1) \varphi_n^-(q_2) dq_1 dq_2 \quad (3.7)$$

ga ega bo'lamiz. (3.7) ga ko'ra $n \geq 3$ da $V^{+-} f\varphi_n^- = 0$ bo'ladi. Shuning uchun $V^{+-} f\varphi_n^-$ ni $n = 1, 2$: da qaraymiz.

$$(V^{+-} f \phi_1^-)(p_1, p_2) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} (\bar{v}(1) + 2\bar{v}(2) \cos p_1 \cos q_1) f(q_1) dq_1 \right] \phi_1^-(p_2), \quad (3.8)$$

$$(V^{+-} f \phi_2^-)(p_1, p_2) = \frac{\bar{v}(2)}{2\pi} \int_{-\pi}^{\pi} f(q_1) dq_1 \phi_2^-(p_2). \quad (3.9)$$

Shunday qilib $\mathfrak{R}_n^{+-}$ fazo $H(k_1, \pi)$ operatorga nisbatan invariant qism fazo tashkil qiladi. Shuningdek $\mathfrak{R}_{n-1}^{+-}$ fazoni ham $H(k_1, \pi)$ operatorga nisbatan invariant qism fazo bo'lishi shu kabi ko'rsatiladi.

3.2 lemmaga ko'ra $H^{+-}(k_1, \pi)$ operatorning $\mathfrak{R}_n^{+-} = L_2^+(T) \otimes L^-(n)$ fazodagi qismi $H_n^{+-}(k_1, \pi), n \in \mathbb{N}$, quyidagi ko'rinishda bo'aladi:

$$H_n^{+-}(k_1, \pi) = [2I + H_0(k_1) - V_n^+] \otimes I^-. \quad (3.10)$$

Bunda $I^- = L^-(n)$ dagi birlik operator, $L_2^+(T)$ fazodagi bir o'lchamli ikki zarrachali $H_n^+(k_1) = 2I + H_0(k_1) - V_n^+$ operator quyidagicha bo'ladi:

$$(H_n^+(k_1) f)(p) = (2 + \varepsilon_{k_1}(p)) f(p) - (V_n^+ f)(p), \quad \varepsilon_{k_1}(p) = 2 - 2 \cos \frac{k_1}{2} \cos p, \quad f \in L_2^+(T).$$

3.2 lemmani isbotiga ko'ra $n \geq 3$, da $V_n^+ = 0$ bo'ladi, $(V_1^+ f)(p_1)$ va $(V_2^+ f)(p_1)$ operatorlar (3.8) va (3.9) tenglikdagi to'rtburchak qavs ichidagi qismi bo'ladi.

$H_{n-1}^{+-}(k_1, \pi)$ operatorni ko'rinishini keltiramiz

$$H_{n-1}^{+-}(k_1, \pi) = H^{+-}(k_1, \pi)|_{\mathfrak{R}_{n-1}^{+-}} := [2I + H_0(k_1) - V_{n-1}^-] \otimes I^+, \quad n \in \mathbb{N}. \quad (3.11)$$

$$(H_0(k_1) - V_0^-) f(p) = \varepsilon_{k_1}(p) f(p) - \frac{1}{\pi} \int_{-\pi}^{\pi} (\bar{v}(1) \sin p \sin q + \bar{v}(2) \sin 2p \sin 2q) f(q) dq,$$

$$(H_0(k_1) - V_1^-) f(p) = \varepsilon_{k_1}(p) f(p) - \frac{\bar{v}(2)}{\pi} \int_{-\pi}^{\pi} \sin p \sin q f(q) dq,$$

$$(H_0(k_1) - V_n^-) f(p) = H_0(k_1) f(p) = \varepsilon_{k_1}(p) f(p), \quad n \geq 2, \quad f \in L_2^-(T).$$

(3.10) va (3.11) tengliklarga ko'ra $H_n^{+-}(k_1, \pi)$ va $H_{n-1}^{+-}(k_1, \pi), n \in \mathbb{I}$ operatorlarni xos qiymatlarini o'rganish masalasi $H_n^+(k_1)$ va $H_{n-1}^-(k_1), n \in \mathbb{I}$ operatorlar xos qiymatini o'rganish masalasiga keladi, ya'ni ikki o'lchamli masala bir o'lchamli masalaga keltiriladi. Shuningdek, $H(\pi, \pi)$ operatorning ikki karrali $z_1(\pi, \pi) = 4 - \bar{v}(1)$ xos qiymatiga mos $\phi_{(1,0)}^{+-}$ va $\phi_{(0,1)}^{+-}$ ((3.3) qarang), xos funksiyalar uchun $\phi_{(1,0)}^{+-} \in \mathfrak{R}_0^{+-}$ va $\phi_{(0,1)}^{+-} \in \mathfrak{R}_1^{+-}$ bo'ladi. Demak, $z_1(\pi, \pi) = 4 - \beta$ xos qiymat $H_0^{+-}(\pi, \pi)$, $H_1^{+-}(\pi, \pi)$ operatorlar uchun oddiy xos qiymat bo'adi. $H(\pi, \pi)$ operatorning to'rt karrali $z_2(\pi, \pi) = 4 - \bar{v}(2)$ xos qiymatiga mos xos $\phi_{(2,0)}^{+-}$, $\phi_{(0,2)}^{+-}$, $\phi_{(1,1)}^{+-}$ va $\phi_{(1,1)}^{+-}$, ((3.4) qarang) funksiyalar va $\phi_{(2,0)}^{+-} \in \mathfrak{R}_0^{+-}$, $\phi_{(0,2)}^{+-} \in \mathfrak{R}_2^{+-}$, $\phi_{(1,1)}^{+-} \in \mathfrak{R}_1^{+-}$ va $\phi_{(1,1)}^{+-} \in \mathfrak{R}_1^{+-}$ lar o'rinni. Bundan, $z_2(\pi, \pi)$ xos qiymat $H_0^{+-}(k_1, \pi)$, $H_2^{+-}(k_1, \pi)$ $H_1^{+-}(k_1, \pi)$ va $H_1^{+-}(k_1, \pi)$ lar uchun oddiy xos qiymat boladi.

4. $H_n^{+-}(k_1)$ opertatorni xos qiymatlari

$H(\mathbf{k})$ operatorning xos funksiyalari H Hamiltonianni bog'langan holatlarini ifodalaydi, xos qiymatlari esa bog'langan holatlar energiyasini ifodalaydi. Shu maqsadda $H(\mathbf{k})$ operatorning xos funksiyalarini o'rgsanamiz, bu esa $H(\mathbf{k})$ operatorni xos qiymatlarini topish masalasiga keladi. Ishda $H(k_1, \pi)$ operatorni xos qiymatlarini o'rganish masalasi $H_n^{+-}(k_1, \pi)$ va $H_{n-1}^{+-}(k_1, \pi), n \in \mathbb{I}$ operatorlarni xos qiymatlarini o'rganish masalasiga keltirildi va so'ngda (3.10) va (3.11) tengliklarga asosan

$$H_n^+(k_1) = 2I + H_0(k_1) - V_n^+, H_{n-1}^-(k_1) = 2I + H_0(k_1) - V_{n-1}^-, n \in \Gamma$$

bir o'ldhamli Shredinger operatorlarni o'rganamiz. Dastlab $H_n^+(k_1)$, $n \in \Gamma$ operatorlarni spektral xossalarini o'rganamiz.

4.1 Lemma. *Ixtiyoriy $n \in \Gamma$ uchun $H_n^+(k_1)$ operatorning muhim spektri*

$$[2 - 2\cos \frac{k_1}{2}, 2 + 2\cos \frac{k_1}{2}] \text{ kesmadan iborat bo'ladi.}$$

4.1 lemmani isboti muhim spektr uchun Veyl teoremasidan kelib chiqadi [6].

4.2 Teorema. *Ixtiyoriy $k_1 \in \Gamma = (-\pi, \pi]$ uchun $H_1^+(k_1)$ operator o'zini muhim spektridan chapda kamida bitta xos qiymatga ega bo'ladi.*

Isbot. Agar $G_1^+(k_1, z)$ ($H_1^+(k_1)$ operatorga mos) Birman-Shvinher operatorini $z < m(k_1, \pi)$ da birdan katta xos qiymatga ega ekanligini ko'rsatsak, $H_1^+(k_1)$ operator ham $z < m(k_1, \pi)$ dan chapda xos qiymat mavjudligi kelib chiqadi.

$G_1^+(k_1, z)$ operatorni musbat va kompaktligidan

$$\sup_{\|\psi\|=1} (G_1^+(k_1, z)\psi, \psi) = \mathbb{E}_1^+(k_1, z) \mathbb{E} \lambda^+(k_1, z)$$

son eng katta xos qiymati bo'ladi. Quyidagi skalyar ko'paytmani qaraymiz:

$$(G_1^+(k_1, z)\psi, \psi) = (r_0(k_1, z)(V_1^+)^{\frac{1}{2}}\psi, (V_1^+)^{\frac{1}{2}}\psi) = \int_{-\pi}^{\pi} \frac{|(V_1^+)^{\frac{1}{2}}\psi(p)|^2}{4 - 2\cos \frac{k_1}{2} \cos p - z} dp. \quad (4.1)$$

Agar (4.1) da $\psi(p)$ xos funksiya o'rniga $\phi_0^+(p) = \frac{1}{\sqrt{2\pi}}$ ni qo'ysak

$$(G_1^+(k_1, z)\phi_0^+, \phi_0^+) = \frac{\bar{v}(2)}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{4 - 2\cos \frac{k_1}{2} \cos p - z}.$$

ga ega bo'lamiz va quyidagi limitga

$$\lim_{z \rightarrow m(k_1)-} (G_1^+(k_1, z)\phi_0^+, \phi_0^+) = +\infty$$

ko'ra $z_n \in (m(k_1) - \delta, m(k_1))$ son mavjud bo'lib,

$$(G_1^+(k_1, z_n)\phi_0^+, \phi_0^+) > 1$$

tengsizlik o'rinli. Bundan esa $G_1^+(k_1, z_n)$ Birman-Shvinger operatori birdan katta xos qiymatga ega ekanligi kelib chiqadi. $H_1^+(\pi) = 2I + H_0(\pi) - V_1^+$ operator ikkita $z_1 = 4 - \bar{v}(1)$ va $z_2 = 4 - \bar{v}(2)$ xos qiymatlarga ega bo'lib, ularga mos normalangan xos funksiyalari $\phi_0^+ = \frac{1}{\sqrt{2\pi}}$ va $\phi_1^+ = \frac{\cos p}{\sqrt{\pi}}$.

$H_n^+(-k_1) = H_n^+(k_1)$ ekanligida $k_1 \in [0, \pi]$ deb olish mumkin. $k_1 = \pi - 2\varepsilon$ belgilash olib $k_1 \in [0, \pi]$ dan $\varepsilon \in [0, \frac{\pi}{2}]$ ga ega bo'lamiz.

4.3 Teorema. *Ixtiyoriy $\varepsilon > 0$ son uchun, shunday $\delta > 0$ mavjudki har bir $\varepsilon \in (0, \delta)$ uchun $H_1^+(\pi - 2\varepsilon)$ operator ikkita oddiy $z_1^+(\varepsilon)$ va $z_2^+(\varepsilon)$ xos qiymatga ega va ular quyidagi asimptotik ko'rinishda bo'ladi:*

$$z_1^+(\varepsilon) = z_1 - \frac{2}{\bar{v}(1) - \bar{v}(2)} \varepsilon^2 + O(\varepsilon^4), \quad \varepsilon \rightarrow 0,$$

$$z_2^+(\varepsilon) = z_2 - \frac{\bar{v}(1) - 3\bar{v}(2)}{\bar{v}(2)(\bar{v}(1) - \bar{v}(2))} \varepsilon^2 + O(\varepsilon^4), \quad \varepsilon \rightarrow 0.$$

Isbot. $H_1^+(\pi - 2\varepsilon)f = zf$ xos qiymat uchun tenglamani qaraymiz. Bu tenglama quyidagi tenglama bilan teng kuchli:

$$(4 - 2\sin \varepsilon \cos p - z)f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\bar{v}(1) + 2\bar{v}(2)\cos p \cos s] f(s) ds.$$

$$\begin{cases} c_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) ds, \\ c_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos s f(s) ds \end{cases} \quad (4.2)$$

almashtirishlarni olib

$$f(p) = \frac{\bar{v}(1)}{4 - 2\sin \varepsilon \cos p - z} c_1 + \frac{\bar{v}(2)\cos p}{4 - 2\sin \varepsilon \cos p - z} c_2 \quad (4.3)$$

ga ega bo'amiz. (4.3) ni (4.2) ga qo'yib c_1 va c_2 ga nisbatan bir jinsli tenglama hosil qilamiz

$$\begin{cases} c_1 = c_1 \frac{\bar{v}(1)}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{4 - 2\sin \varepsilon \cos p - z} + c_2 \frac{2\bar{v}(2)}{2\pi} \int_{-\pi}^{\pi} \frac{\cos p dp}{4 - 2\sin \varepsilon \cos p - z}, \\ c_2 = c_1 \frac{\bar{v}(1)}{2\pi} \int_{-\pi}^{\pi} \frac{\cos p dp}{4 - 2\sin \varepsilon \cos p - z} + c_2 \frac{2\bar{v}(2)}{2\pi} \int_{-\pi}^{\pi} \frac{\cos^2 p dp}{4 - 2\sin \varepsilon \cos p - z}. \end{cases}$$

Bu tenglama

$$\Delta(\varepsilon, z) = \begin{vmatrix} 1 - \bar{v}(1)\Delta_{11}(\varepsilon, z) & -2\bar{v}(2)\Delta_{12}(\varepsilon, z) \\ -\bar{v}(1)\Delta_{12}(\varepsilon, z) & 1 - 2\bar{v}(2)\Delta_{22}(\varepsilon, z) \end{vmatrix}$$

$$= (1 - \bar{v}(1)\Delta_{11}(\varepsilon, z))(1 - 2\bar{v}(2)\Delta_{22}(\varepsilon, z)) - 2\bar{v}(1)\bar{v}(2)\Delta_{12}^2(\varepsilon, z) \quad (4.4)$$

determinant nolga teng bo'lgandagina yechimga ega bo'ladi.

$$\Delta_{11}(\varepsilon, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dp}{4 - z - 2\sin \varepsilon \cos p}, \quad \Delta_{12}(\varepsilon, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos p dp}{4 - z - 2\sin \varepsilon \cos p},$$

$$\Delta_{22}(\varepsilon, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos^2 p dp}{4 - z - 2\sin \varepsilon \cos p}.$$

Kichik ε da $1/(4 - z - 2\sin \varepsilon \cos p)$ funksiyani quyidagi qator ko'rinishda yozish mumkin.

$$\frac{1}{(4 - z)(1 - \frac{2\sin \varepsilon \cos p}{4 - z})} =$$

$$= \frac{1}{4 - z} \left(1 + \frac{2\sin \varepsilon \cos p}{4 - z} + \frac{4\sin^2 \varepsilon \cos^2 p}{(4 - z)^2} + \dots + \frac{(2\sin \varepsilon \cos p)^n}{(4 - z)^n} + \dots \right). \quad (4.5)$$

(4.5) yoyilmadan foydalanib, Δ_{11}, Δ_{12} va Δ_{22} funksiyalarni ε^2 aniqligida hisoblaymiz.

$$\Delta(\varepsilon, z) = 1 - \frac{\bar{v}(1) + \bar{v}(2)}{4 - z} + \frac{\bar{v}(1)\bar{v}(2)}{(4 - z)^2} + \frac{2\bar{v}(1) + 3\bar{v}(2)}{(4 - z)^3} \sin^2 \varepsilon + \frac{3\bar{v}(1)\bar{v}(2)}{(4 - z)^4} \sin^2 \varepsilon + O(\varepsilon^4).$$

Qiyin bo'lmagan hisoblashlardan so'ng $\Delta(\varepsilon, z)$ funksiyani nollarini $o(\varepsilon^2)$ aniqlikda topamiz

$$z_0(\varepsilon) = 4 - \bar{v}(1) - \frac{2}{\bar{v}(1) - \bar{v}(2)} \sin^2 \varepsilon, \quad z_1(\varepsilon) = 4 - \bar{v}(2) - \frac{\bar{v}(1) - 3\bar{v}(2)}{\bar{v}(2)(\bar{v}(1) - \bar{v}(2))} \sin^2 \varepsilon.$$

$\varepsilon \rightarrow 0$ da $\sin \varepsilon$: ε ekanligidan teorema tasdig'i kelib chiqadi.

4.4 Teorema. Har bir $k_1 \in T$ uchun $H_2^+(k_1)$ operator yagona xos

$$z_2^+(k_1) = 4 - \sqrt{\bar{v}^2(2) + 4\cos^2 \frac{k_1}{2}}$$

qiymatga ega bo'lib, unga mos xos funksiyasi

$$f_2^+(p) = \frac{C}{4 - 2\cos \frac{k_1}{2} \cos p - z_2^+(k_1)} \in L_2^+(\Gamma)$$

bo'ladi, bunda C o'zgarmas son.

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К ИССЛЕДОВАНИЮ ОДНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА ПАРАБОЛО-ГИПЕРБОЛИЧЕСКОГО ТИПА В СМЕШАННОЙ ПЯТИУГОЛЬНОЙ ОБЛАСТИ С ТРЕМЯ ЛИНИЯМИ ИЗМЕНЕНИЯ ТИПА

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Аннотация. Настоящая работа посвящена постановке и исследованию одной краевой задачи для параболо-гиперболического уравнения четвертого порядка в смешанной области с тремя линиями изменения типа, когда одна из характеристик операторов первого порядка параллельна оси ординат, а угловой коэффициент другой характеристики равен минус единице. Доказана теорема об однозначной разрешимости этой поставленной задачи. В ходе доказательства этой теоремы применены методы построения решения, дифференциальных и интегральных уравнений, а также метод продолжения.

Ключевые слова. дифференциальные и интегральные уравнения, метод построения решения, метод продолжения, краевая задача, параболо-гиперболический тип, однозначная разрешимость.

TO THE STUDY OF A BOUNDARY VALUE PROBLEM FOR A FOURTH-ORDER EQUATION OF PARABOLIC-HYPERBOLIC TYPE IN A MIXED PENTAGONAL DOMAIN WITH THREE LINES OF TYPE CHANGE

Abstract. This paper is devoted to the formulation and study of a boundary value problem for a fourth-order parabolic-hyperbolic equation in a mixed domain with three lines of type change, where one characteristic of the first-order operators is parallel to the ordinate axis, and the slope of the other characteristic is equal to minus one. A theorem on the unique solvability of this problem is proved. The proof of this theorem utilizes solution construction methods, differential and integral equations, and the continuation method.

Keywords: differential and integral equations, solution construction method, continuation method, boundary value problem, parabolic-hyperbolic type, unique solvability.

UCHTA TIP O‘ZGARISH CHIZIG‘IGA EGA BO‘LGAN ARALASH BESH BURCHAKLI SOHADA TO‘RTINCHI TARTIBLI PARABOLIK-GIPERBOLIK TIPDAGI TENGLAMA UCHUN CHEGARAVIY MASALANI O‘RGANISH HAQIDA

Annotatsiya. Ushbu maqola birinchi tartibli operatorlarning bir xarakteristikasi ordinata o‘qiga parallel, ikkinchi xarakteristikaning burchak koeffitsiyenti esa minus birga teng bo‘lgan uchta turdagi o‘zgarish chizig‘iga ega aralash sohada to‘rtinchi tartibli parabol-giperbolik tipdagi tenglama uchun chegaraviy masalani qo‘yish va o‘rganishga bag‘ishlangan. Ushbu masala yechimining mavjud va yagonaligi haqidagi teorema isbotlangan. Ushbu teoremaning isboti yechimni qurish, differensial va integral tenglamalar hamda davom etish usullaridan foydalanadi.

Kalit so‘zlar: differensial va integral tenglamalar, yechimni qurish usuli, davom ettirish usuli, chegaraviy masala, parabol-giperbolik tip, yagona yechilishi.

Введение. Фундаментальное исследование уравнений эллиптического-гиперболического типа второго порядка начал изучать итальянский математик Трикоми в 20-ые годы прошлого века [1]. После этого начали изучать многие разные задачи для уравнений таких типов. Обзор теоретических и прикладных исследований приведен в работах и книгах А.В.Бицадзе [2], Л.Берса [3], М.С.Салохитдинова [4], Т.Д.Джураева [5] и т.д.

1959 году И.М.Гельфанд [6] приводит пример, связанный с движением газа в канале, окруженной пористой средой: в канале движение газа описывается волновым уравнением, вне его – уравнением диффузии. Затем в 70-80-годов прошлого века начали изучать различные задачи для уравнений третьего и высокого порядков парабола-гиперболического типа. Такие задачи исследованы, в основном, Т.Д.Джураевым и его учениками (например, см. [7], [8]). В настоящее время изучение различных краевых задач для уравнений третьего и высокого порядков парабола-гиперболического типа развивается в широком плане (например, см. [9]-[18]).

Постановка задачи. Здесь рассматривается одна краевая задача для следующего уравнения четвертого порядка вида

$$\frac{\partial}{\partial y} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (Lu) = 0, \quad (1)$$

в области G , где $G = G_1 \cup G_2 \cup G_3 \cup G_4 \cup J_1 \cup J_2 \cup J_3$, G_1 – прямоугольник с вершинами в точках $A(0;0)$, $B(1;0)$, $B_0(1,1)$, $A_0(0,1)$; G_2 – треугольник с вершинами в точках $D(-1,0)$, $E(1/2, -3/2)$, $C(2,0)$; G_3 и G_4 – прямоугольники с вершинами в точках A , D , $D_0(-1,1)$, A_0 и B , B_0 , $C_0(2,1)$, $C(2,0)$ соответственно; J_1 , J_2 и J_3 – открытые отрезки с вершинами в точках C , D ; A , A_0 и B , B_0 соответственно, а

$$Lu = \begin{cases} u_{xx} - u_y, & (x, y) \in G_1, \\ u_{xx} - u_{yy}, & (x, y) \in G_j, \quad j = 2, 3, 4. \end{cases}$$

Для уравнения (1) ставится следующая задача:

Задача 1. Найти функцию $u(x, y)$, которая

1) непрерывна в $\bar{G}$ и в области $G \setminus J_1 \setminus J_2 \setminus J_3$ имеет непрерывные производные, участвующие в уравнение (1), причем u_x , u_y , u_{xx} , u_{xy} , u_{yy} – непрерывны в G вплоть до части границы области G , указанные в краевых условиях;

2) удовлетворяет уравнению (1) в области $G \setminus J_1 \setminus J_2 \setminus J_3$;

3) удовлетворяет следующим краевым условиям:

$$u(2, y) = \varphi_1(y), \quad 0 \leq y \leq 1; \quad (2)$$

$$u(-1, y) = \varphi_2(y), \quad 0 \leq y \leq 1; \quad (3)$$

$$u_x(2, y) = \varphi_3(y), \quad 0 \leq y \leq 1; \quad (4)$$

$$u|_{CE} = \psi_1(x), \quad 1/2 \leq x \leq 2; \quad (5)$$

$$u|_{DP} = \psi_2(x), \quad -1 \leq x \leq -1/2; \quad (6)$$

$$u|_{QE} = \psi_3(x), \quad 0 \leq x \leq 1/2; \quad (7)$$

$$\left. \frac{\partial u}{\partial n} \right|_{DE} = \psi_4(x), \quad -1 \leq x \leq 1/2; \quad (8)$$

$$\left. \frac{\partial u}{\partial n} \right|_{CE} = \psi_6(x), \quad 1/2 \leq x \leq 2; \quad (9)$$

$$\left. \frac{\partial^2 u}{\partial n^2} \right|_{CE} = \psi_7(x), \quad 1/2 \leq x \leq 2; \quad (10)$$

4) удовлетворяет следующим условиям склеивания:

$$u(x, +0) = u(x, -0) = T(x), \quad -1 \leq x \leq 1; \quad (11)$$

$$u_y(x, +0) = u_y(x, -0) = N(x), \quad -1 \leq x \leq 1; \quad (12)$$

$$u_{yy}(x, +0) = u_{yy}(x, -0) = M(x), \quad -1 < x < 2; \quad (13)$$

$$u_{yyy}(x, +0) = u_{yyy}(x, -0) = \Theta(x), \quad -1 < x < 2; \quad (14)$$

$$u(+0, y) = u(-0, y) = \tau_4(y), \quad 0 \leq y \leq 1; \quad (15)$$

$$u_x(+0, y) = u_x(-0, y) = \nu_4(y), \quad 0 \leq y \leq 1; \quad (16)$$

$$u_{xx}(+0, y) = u_{xx}(-0, y) = \mu_4(y), \quad 0 < y \leq 1; \quad (17)$$

$$u(1+0, y) = u(1-0, y) = \tau_5(y), \quad 0 \leq y \leq 1; \quad (18)$$

$$u_x(1+0, y) = u_x(1-0, y) = \nu_5(y), \quad 0 \leq y \leq 1; \quad (19)$$

$$u_{xx}(1+0, y) = u_{xx}(1-0, y) = \mu_5(y), \quad 0 < y \leq 1, \quad (20)$$

где $\varphi_i (i = \overline{1, 3})$, $\psi_j (j = \overline{1, 4})$, ψ_6, ψ_7 – заданные достаточно гладкие функции, n – внутренняя нормаль к прямой $x + y = -1$ или $x - y = 2$, а $Q(0, -1)$, $P(-1/2, -1/2)$,

$$T(x) = \begin{cases} \tau_1(x), & \text{if } 0 \leq x \leq 1, \\ \tau_2(x), & \text{if } -1 \leq x \leq 0, \\ \tau_3(x), & \text{if } 1 \leq x \leq 2; \end{cases} \quad N(x) = \begin{cases} \nu_1(x), & \text{if } 0 \leq x \leq 1, \\ \nu_2(x), & \text{if } -1 \leq x \leq 0, \\ \nu_3(x), & \text{if } 1 \leq x \leq 2; \end{cases}$$

$$M(x) = \begin{cases} \mu_1(x), & \text{if } 0 < x < 1, \\ \mu_2(x), & \text{if } -1 < x < 0, \\ \mu_3(x), & \text{if } 1 < x < 2; \end{cases} \quad \Theta(x) = \begin{cases} \theta_1(x), & \text{if } 0 < x < 1, \\ \theta_2(x), & \text{if } -1 < x < 0, \\ \theta_3(x), & \text{if } 1 < x < 2, \end{cases}$$

$\tau_i, \nu_i, \mu_i (i = \overline{1, 5})$, $\theta_1, \theta_2, \theta_3$ – неизвестные функции.

Исследование задачи. Теорема. Если $\varphi_1, \varphi_2 \in C^4[0, 1]$, $\varphi_3 \in C^3[0, 1]$, $\psi_1 \in C^4[1/2, 2]$, $\psi_2 \in C^4[-1, -1/2]$, $\psi_3 \in C^4[0, 1/2]$, $\psi_4 \in C^3[-1, 1/2]$, $\psi_6 \in C^3[1/2, 2]$, $\psi_7 \in C^2[1/2, 2]$, причем выполняется условие согласования $\tau_2(-1) = \psi_2(-1) = \varphi_2(0)$, $\psi_1(1/2) = \psi_2(1/2)$, $\tau_1(0) = \tau_2(0)$, $\tau_1(1) = \tau_3(1)$, $\tau_1'(1) = \tau_3'(1)$, $\tau_1'(0) = \tau_2'(0)$, $\tau_3(2) = \varphi_1(0) = \psi_1(2)$, $\psi_4'(1/2) = -\psi_6'(1/2)$, то задача-1 допускает единственное решение.

Доказательство. Теорему докажем методом построения решения. Для этого уравнение (1) перепишем в виде

$$u_{1xx} - u_{1y} = \omega_{11}(x) + \omega_{12}(x + y), \quad (x, y) \in G_1, \quad (21)$$

$$u_{ixx} - u_{iyy} = \omega_{i1}(x) + \omega_{i2}(x + y), \quad (x, y) \in G_i \quad (i = 2, 3, 4), \quad (22)$$

где $u(x, y) = u_i(x, y)$, $(x, y) \in G_i$ ($i = \overline{1, 4}$), причем $\omega_{i1}(x)$, $\omega_{i2}(x + y)$ ($i = \overline{1, 4}$) – неизвестные функции.

Исследуем задачу сначала в области G_2 . Решение уравнения (22) ($i = 2$), удовлетворяющее условиям (11), (12) представляется в виде

$$u_2(x, y) = \frac{1}{2} [T(x + y) + T(x - y)] + \frac{1}{2} \int_{x-y}^{x+y} N(t) dt - \frac{1}{2} \int_0^y d\eta \int_{x-y+\eta}^{x+y-\eta} [\omega_{21}(\xi) + \omega_{22}(\xi + \eta)] d\xi. \quad (23)$$

Подставляя (23) в условия (9) и (10) после длинных вычислений и упрощений, находим

$$\omega_{21}(x) = 2[\psi_7(x) - \psi_7(2)] + \omega_{21}(2), \quad 1/2 \leq x \leq 2. \quad (24)$$

$$\omega_{22}(x + y) = -\sqrt{2}\psi_6'\left(\frac{x + y + 2}{2}\right) - 2\left[\psi_7\left(\frac{x + y + 2}{2}\right) - \psi_7(2)\right] - \omega_{21}(2), \quad -1 \leq x + y \leq 2. \quad (25)$$

Слагая (24) и (25), имеем

$$\begin{aligned} \omega_{21}(x) + \omega_{22}(x+y) = & -\sqrt{2}\psi'_6\left(\frac{x+y+2}{2}\right) + \\ & + 2\left[\psi_7(x) - \psi_7\left(\frac{x+y+2}{2}\right)\right], 1/2 \leq x \leq 2, -1 \leq x+y \leq 2. \end{aligned} \quad (26)$$

Далее, подставляя (23) в условие (8) и дифференцируя полученное равенство, находим

$$\omega_{21}(x) = \sqrt{2}\psi'_4(x) - \omega_{22}(-1), -1 \leq x \leq 1/2. \quad (27)$$

Теперь пользуясь из непрерывности функции $\omega_{21}(x)$ при $x = 1/2$, слагая (25) при $-1 \leq x+y \leq 1/2$ и (27) ($-1 \leq x \leq 1/2$), находим

$$\begin{aligned} \omega_{21}(x) + \omega_{22}(x+y) = & \sqrt{2}[\psi'_4(x) - \psi'_4(1/2)] - \\ & - 2\left[\psi_7\left(\frac{x+y+2}{2}\right) - \psi_7\left(\frac{1}{2}\right)\right] - \sqrt{2}\psi'_6\left(\frac{x+y+2}{2}\right), \\ & -1 \leq x \leq 1/2, -1 \leq x+y \leq 1/2. \end{aligned} \quad (28)$$

Таким образом, мы нашли функцию $\omega_{21}(x) + \omega_{22}(x+y)$. Она дается по формулам (26) и (28).

Теперь подставляя (23) в (5), получим соотношение

$$T'(x) + N(x) = \alpha_1(x), -1 \leq x \leq 2, \quad (29)$$

где $\alpha_1(x)$ – известная функция.

а) При $0 \leq x \leq 1$ уравнение (29) имеет вид

$$\tau'_1(x) + \nu_1(x) = \alpha_1(x), 0 \leq x \leq 1, \quad (30)$$

б) при $-1 \leq x \leq 0$ –

$$\tau'_2(x) + \nu_2(x) = \alpha_1(x), -1 \leq x \leq 0, \quad (31)$$

в) а при $1 \leq x \leq 2$ –

$$\tau'_3(x) + \nu_3(x) = \alpha_1(x), 1 \leq x \leq 2. \quad (32)$$

Далее, подставляя (23) в (6), получим соотношение

$$\tau'_2(x) - \nu_2(x) = \delta_1(x), -1 \leq x \leq 0, \quad (33)$$

где $\delta_1(x)$ – известная функция.

А подставляя (23) в (7), имеем

$$\tau'_3(x) - \nu_3(x) = \delta_2(x), 1 \leq x \leq 2, \quad (34)$$

где $\delta_2(x)$ – известная функция.

Из (31) и (33) находим функции $\tau'_2(x)$ и $\nu_2(x)$:

$$\tau'_2(x) = \frac{1}{2}[\alpha_1(x) + \delta_1(x)], \nu_2(x) = \frac{1}{2}[\delta_1(x) - \alpha_1(x)]. \quad (35)$$

Интегрируя первое из (35) от -1 до x , находим

$$\tau_2(x) = \frac{1}{2} \int_{-1}^x [\alpha_1(t) + \delta_1(t)] dt + \psi_2(-1).$$

Далее, из (32) и (34) находим функции $\tau'_3(x)$ и $\nu_3(x)$:

$$\tau'_3(x) = \frac{1}{2}[\alpha_1(x) + \delta_2(x)], \nu_3(x) = \frac{1}{2}[\delta_2(x) - \alpha_1(x)]. \quad (36)$$

Интегрируя первое из (36) от 2 до x , находим

$$\tau_3(x) = \frac{1}{2} \int_2^x [\alpha_1(t) + \delta_2(t)] dt + \psi_1(2).$$

Теперь переходя в уравнении (22) ($i = 2$), к пределу при $y \rightarrow 0$, в силу (11) и (13) получим соотношение

$$\mu_1(x) = \tau_1''(x) - \omega_{21}(x) - \omega_{22}(x). \quad (37)$$

Дифференцируя уравнение (22) ($i = 2$) по y и устремляя y к нулю, получим еще одно соотношение

$$\theta_1(x) = \nu_1''(x) - \omega_{22}'(x). \quad (38)$$

В уравнение (1) устремляя y к нулю, получим соотношение

$$-\nu_1'''(x) + \mu_1''(x) + \mu_1'(x) - \theta_1(x) = 0.$$

Подставляя (30), (37) и (38) в последнее уравнение, затем интегрируя полученное уравнение трижды от 0 до x , имеем

$$\tau_1'(x) + \tau_1(x) = \alpha_2(x) + k_1 \frac{x^2}{2} + k_2 x + k_3, \quad 0 \leq x \leq 1,$$

где $\alpha_2(x)$ – известная функция, а k_1, k_2, k_3 – неизвестные пока постоянные.

Теперь решая последнее уравнение при условиях $\tau_1(0) = \tau_2(0)$, $\tau_1'(0) = \tau_2'(0)$, $\tau_1(1) = \tau_3(1)$, $\tau_1'(1) = \tau_3'(1)$, находим функцию $\tau_1(x)$:

$$\tau_1(x) = \int_0^x e^{t-x} \alpha_2(t) dt + k_1 \left(\frac{x^2}{2} - x + 1 - e^{-x} \right) + k_2 (x - 1 + e^{-x}) + k_3 (1 - e^{-x}) + k_4 e^{-x},$$

где k_1, k_2, k_3, k_4 – уже известные постоянные.

Тогда будут известными и функции $\nu_1(x)$, $\mu_1(x)$, $u_2(x, y)$.

Теперь переходим в область G_4 . Пользуясь условиями склеиваний (11), (12), (13), (14), после некоторых преобразований и вычислений, получим

$$\omega_{41}(x) + \overline{\omega}_{42}(x+y) = \omega_{21}(x) + \omega_{22}(x+y), \quad 1 \leq x+y \leq 2.$$

Далее, в силу условий склеивания на линии BB_0 , после некоторых выкладок, имеем

$$\overline{\omega}_{12}(x+y) = [\tau_5''(x+y-1) - \tau_5'(x+y-1)] + [\omega_{41}(1) + \overline{\omega}_{42}(x+y)] - \omega_{11}(1), \quad 1 \leq x+y \leq 2. \quad (39)$$

На линии изменения типа BC ($1 \leq x \leq 2$) пользуясь условиям склеивания, получим

$$\overline{\omega}_{12}(x+y) = \gamma_1(x+y) - \nu_1'(0) + \overline{\omega}_{12}(0), \quad (40)$$

$$\omega_{11}(x) = \gamma_2(x) + \nu_1'(0) - \overline{\omega}_{12}(0), \quad (41)$$

где $\gamma_1(x+y)$, $\gamma_2(x)$ – известные функции.

Полагая в (41) $x = 1$ и подставляя найденное значение $\omega_{11}(1)$ в (39), находим

$$\overline{\omega}_{12}(x+y) = [\tau_5''(x+y-1) - \tau_5'(x+y-1)] + \gamma_3(x, y), \quad 1 \leq x+y \leq 2, \quad 0 \leq x \leq 1, \quad (42)$$

где $\gamma_3(x, y)$ – известная функция. Теперь сначала рассмотрим следующую задачу:

$$\begin{cases} u_{4xx} - u_{4yy} = \Omega_{41}(x) + \Omega_{42}(x+y), \\ u_4(x, 0) = T_3(x), \quad u_{4y}(x, 0) = N_3(x), \quad 0 \leq x \leq 3, \\ u_4(2, y) = \varphi_1(y), \quad u_{4x}(2, y) = \varphi_3(y), \\ u_{4x}(1, y) = \nu_5(y), \quad u_4(1, y) = \tau_5(y), \quad 0 \leq y \leq 1. \end{cases}$$

Применив метод продолжения, находим решение этой задачи, удовлетворяющее всем условиям, кроме условия $u_{4x}(1, y) = \nu_5(y)$. Дифференцируя это решение и полагая в полученном равенстве $x = 1$ в силу условия $u_{4x}(1, y) = \nu_5(y)$ после некоторых вычислений, получим

$$\nu_5(y) = -\tau'_5(y) + \beta_2(y), \quad (43)$$

где $\beta_2(y)$ – известная функция. Теперь рассмотрим задачу в области G_3 . Пользуясь условиям склеивания (11), (12), (13), (14) при $-1 \leq x \leq 0$, после некоторых преобразований и упрощений, находим

$$\begin{aligned} \omega_{31}(x) &= \omega_{21}(x) + \omega_{22}(0) - \omega_{32}(0), \quad -1 \leq x \leq 0, \\ \bar{\omega}_{32}(x+y) &= \omega_{22}(x+y) - \omega_{22}(0) + \omega_{32}(0), \quad -1 \leq x+y \leq 0. \end{aligned}$$

Теперь сначала рассмотрим следующую вспомогательную задачу:

$$\begin{cases} u_{3xx} - u_{3yy} = \Omega_{31}(x) + \Omega_{32}(x+y), \\ u_3(x, 0) = T_2(x), \quad u_{3y}(x, 0) = N_2(x), \quad -2 \leq x \leq 1, \\ u_3(-1, y) = \varphi_2(y), \quad u_{3x}(0, y) = \nu_4(y), \\ u_3(0, y) = \tau_4(y), \quad 0 \leq y \leq 1. \end{cases}$$

Пользуясь методом продолжения, находим решение этой задачи, удовлетворяющее всем условиям, кроме условия $u_{3x}(0, y) = \nu_4(y)$. Дифференцируя это решение по x и полагая в полученном равенстве $x \rightarrow 0$, после некоторых вычислений и упрощений, получим

$$\nu_4(y) = \int_0^y \tau'_4(\eta) d\eta + \beta_1(y), \quad 0 \leq y \leq 1, \quad (44)$$

где $\beta_1(y)$ – известная функция.

Теперь переходим в область G_1 . Записывая решение уравнения (21), удовлетворяющего условиям (11), (15), (18), и дифференцируя это решение по x , затем в полученном равенстве устремляя x к нулю, с учетом равенства (44) после некоторых вычислений, получим интегральное уравнение типа Абеля. Применяя обращение Абеля к полученному уравнению, приходим к интегральному уравнению Вольтерра второго рода относительно $\tau'_4(y)$:

$$\tau'_4(y) + \int_0^y K_1(y, \eta) \tau'_4(\eta) d\eta + \int_0^y K_2(y, \eta) \tau'_5(\eta) d\eta = g_1(y), \quad 0 \leq y \leq 1, \quad (45)$$

где $K_1(y, \eta)$, $K_2(y, \eta)$, $g_1(y)$ – известные функции.

Затем в полученном равенстве для $u_{1x}(x, y)$ устремляя x к единице, с учетом равенства (43), получим интегральное уравнение Вольтерра второго рода относительно $\tau'_5(y)$:

$$\tau'_5(y) + \int_0^y K_3(y, \eta) \tau'_5(\eta) d\eta + \int_0^y K_4(y, \eta) \tau'_4(\eta) d\eta = g_2(y), \quad 0 \leq y \leq 1, \quad (46)$$

где $K_3(y, \eta)$, $K_4(y, \eta)$, $g_2(y)$ – известные функции.

Решая систему уравнений (45), (46), находим функции $\tau'_4(y)$ и $\tau'_5(y)$, тем самым и функции $\tau_4(y)$, $\tau_5(y)$, $\nu_4(y)$, $\nu_5(y)$, $\omega_{11}(x) + \bar{\omega}_{12}(x+y)$, $T_2(x)$, $T_3(x)$, $\omega_{41}(x) + \bar{\omega}_{32}(x+y)$. Тогда будут известными и функции $u_3(x, y)$, $u_4(x, y)$ и $u_1(x, y)$.

Замечание. Если вместо условия (5), (6) и (7) взять условия

$$u|_{DE} = \psi_1(x), \quad -1 \leq x \leq 1/2; \quad u|_{SE} = \psi_2(x), \quad 1/2 \leq x \leq 1;$$

$$u|_{RC} = \psi_3(x), \quad 3/2 \leq x \leq 2 \text{ соответственно, получим равенство}$$

$$\alpha_1'''(x) - \omega'_{21}(x) - \omega''_{21}(x) - \omega''_{22}(x) = 0, \quad 0 \leq x \leq 1, \text{ то есть невозможно получить уравнение}$$

относительно неизвестной функции $\tau_1(x)$. В этом случае такая постановка задачи 1 некорректна при

$$\gamma_2 = -1.$$

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О РЕГУЛЯРИЗАЦИИ ЗАДАЧИ МАЛЫХ РАДИАЛЬНЫХ КОЛЕБАНИЙ

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Аннотация. В работе рассматривается задача Дирихле для волнового уравнения, относящаяся к классу некорректно поставленных задач. Предложен метод регуляризации решения на основе спектрального разложения по собственным функциям соответствующего оператора. Получены оценки, обеспечивающие сходимость регуляризованных решений к точному решению при определённых условиях на параметры задачи. Особое внимание уделено случаю, когда параметры связаны с алгебраическими числами, что позволяет использовать диофантовые оценки для устранения эффекта малых знаменателей. Исследована устойчивость решения при наличии погрешностей в исходных данных и доказана равномерная сходимость по пространственной и временной переменным.

Ключевые слова: задача Дирихле, волновое уравнение, регуляризация, некорректные задачи, малые знаменатели, спектральное разложение, устойчивость решения, сходимость, алгебраические числа, диофантовые оценки.

ON THE REGULARIZATION OF THE PROBLEM OF SMALL RADIAL OSCILLATIONS

Abstract. This paper considers the Dirichlet problem for the wave equation, which belongs to the class of ill-posed problems. A regularization method based on spectral decomposition in terms of eigenfunctions of the corresponding operator is proposed. Estimates ensuring the convergence of regularized solutions to the exact solution under certain conditions on the problem parameters are obtained. Special attention is given to the case when the parameters are related to algebraic numbers, which allows the use of Diophantine estimates to eliminate the effect of small denominators. The stability of the solution under data perturbations is studied, and uniform convergence with respect to spatial and temporal variables is proved.

Keywords: Dirichlet problem, wave equation, regularization, ill-posed problems, small denominators, spectral decomposition, stability of solution, convergence, algebraic numbers, Diophantine estimates.

KICHIK RADIAL TEBRANISHLAR MASALASINI REGULARIZATSIYA HAQIDA

Annotatsiya. Ushbu ishda to'liq tenglamasi uchun Dirixle masalasi ko'rib chiqiladi, u noto'g'ri qo'yilgan masalalar sinfiga kiradi. Mos operatorning xos funksiyalari bo'yicha spektral yoyilishga asoslangan yechimni regularizatsiya qilish usuli taklif etiladi. Masala parametrlariga qo'yilgan ma'lum shartlarda regularizatsiyalangan yechimlarning aniq yechimga yaqinlashishini ta'minlovchi baholar olingan. Parametrlar algebraik sonlar bilan bog'liq bo'lgan holga alohida e'tibor qaratilgan bo'lib, bu kichik maxrajlar muammosini bartaraf etishda Diofant baholaridan foydalanish imkonini beradi. Boshlang'ich ma'lumotlardagi xatoliklar mavjud bo'lganda yechimning barqarorligi o'rganilgan va fazoviy hamda vaqt o'zgaruvchilari bo'yicha tekis yaqinlashuv isbotlangan.

Kalit so'zlar: Dirixle masalasi, to'liq tenglamasi, regularizatsiya, noto'g'ri qo'yilgan masalalar, kichik maxrajlar, spektral yoyilish, yechim barqarorligi, yaqinlashuv, algebraik sonlar, Diofant baholari.

Введение. Краевые задачи для гиперболических уравнений, в частности задача Дирихле для волнового уравнения, занимают важное место в теории уравнений в частных производных и математической физике. Они возникают при моделировании колебательных процессов в сплошных средах, задачах акустики, теории упругости и других прикладных областях. Классические основы

корректности постановки задач, включающие существование, единственность и устойчивость решения, были сформулированы в работах Жак Адамар [1].

Известно, что задача Дирихле для волнового уравнения при задании данных на всей границе области может быть некорректно поставленной. Это связано с явлением малых знаменателей и высокой чувствительностью решения к погрешностям исходных данных, что приводит к потере устойчивости. Подобные эффекты активно исследуются в рамках спектральной теории и анализа рядов Фурье [2, 3].

Для стабилизации решений некорректных задач применяются методы регуляризации, позволяющие строить устойчивые приближённые решения. Основы теории регуляризации были заложены в работах Андрей Тихонов [4], а дальнейшее развитие получили в исследованиях Михаил Лаврентьев и их научной школы [5].

В последние годы наблюдается значительный интерес к развитию методов регуляризации с использованием спектральных разложений, функционального анализа и численных подходов. Современные исследования посвящены как теоретическим аспектам устойчивости, так и построению эффективных алгоритмов для решения некорректных задач (см., например, [6–10]). В частности, изучаются вопросы регуляризации гиперболических задач, устойчивости решений при наличии шумов в данных и сходимости регуляризованных аппроксимаций.

Особый интерес представляет анализ задач, в которых возникают малые знаменатели, связанные с диофантовыми свойствами параметров. Использование результатов теории трансцендентных чисел, включая оценки типа теоремы Лиувилля ([11]), позволяет получить необходимые условия сходимости спектральных разложений.

В настоящей работе рассматривается задача Дирихле для волнового уравнения, предлагается метод регуляризации её решения и доказывается сходимость регуляризованных решений к точному решению. Получены оценки устойчивости в случае неточных исходных данных и установлены условия равномерной сходимости по пространственной и временной переменным.

Постановка задачи. Рассмотрим радиальные колебания шара, при которых каждая точки шара смещаются только по радиусу. В этом случае колебательный процесс, который начался из состояния покоя, описывается следующим образом [12]:

$$u_{tt} = \Delta u, \quad 0 < t < T, |x| < R. \quad (1)$$

с условиями

$$u|_{t=0} = 0, \quad u|_{t=T} = \psi(r), \quad |x| = r \leq R, \quad (2)$$

$$u|_{|x|=R} = 0, \quad 0 \leq t \leq T. \quad (3)$$

Решая соответствующую спектральную задачу, получим, что последовательность собственных значений имеет вид $\lambda_k = \left(\frac{\pi k}{R}\right)^2$, $k = 1, 2, \dots$, а полная ортонормированная система собственных функций описывается следующим образом:

$$v_k(r) = \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right), \quad k = 1, 2, \dots \quad (4)$$

Тогда решение задачи (1)-(3) имеет вид

$$u(r, t) = \sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k}{R} t\right)}{\sin\left(\frac{\pi k}{R} T\right)} \psi_k v_k(r), \quad (5)$$

где ψ_k — коэффициенты разложения функции $\psi(r)$ в ряд Фурье по системе функций (4).

Разрешимость задачи. Итак, разложив решение задачи по собственным функциям (4), мы получили решение в виде (5). Имеет место следующая теорема.

Теорема 1. Пусть число $\frac{T}{R}$ является алгебраическим числом степени 2. Если $\psi \in H^5(|x| < R)$ то функция, представленная формулой (5), является решением задачи (1)-(3).

Доказательство. Сначала напомним, что

$$H^\alpha(\Omega) = \{f \in L_2(\Omega) \mid \sum_{k=1}^{\infty} |(f, v_k)|^2 k^{2\alpha} < \infty\}.$$

Осталось доказать, что ряд (5), а также производные от этого решения, участвующие в уравнении (1), сходятся равномерно. Вторая производная по времени решения (5), имеет вид

$$u_{tt}(r, t) = -\left(\frac{\pi}{R}\right)^2 \sum_{k=1}^{\infty} \frac{k^2 \psi_k \sin\left(\frac{\pi k}{R} t\right)}{\sin\left(\frac{\pi k}{R} T\right)} \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right) \quad (6)$$

Достаточно доказать сходимость ряда (6). Соответствующий мажорантный ряд имеет вид:

$$|u_{tt}(r, t)| \leq C \sum_{k=1}^{\infty} \frac{k^3 |\psi_k|}{|\sin(\lambda_k T)|}, \quad (7)$$

т.к. $\sin\left(\frac{\pi k}{R}r\right)$ ведет себя как $\frac{\pi k}{R}r$ при $r \rightarrow 0$.

Оценим знаменатель ряда (7). Напомним

$$\sin\left(\frac{\pi k}{R}T\right) = 0$$

$$\frac{\pi k}{R}T = \pi m.$$

Введем обозначение

$$\delta_{mk} = \frac{\pi k}{R}T - \pi m$$

или

$$\frac{\pi k}{R}T = \delta_{mk} + \pi m. \quad (8)$$

Отметим, что можно считать, что $|\delta_{mk}| < \frac{\pi}{2}$.

Из (8) находим

$$\frac{T}{R} = \frac{\delta_{mk}}{\pi k} + \frac{m}{k}. \quad (9)$$

По условию теоремы $\frac{T}{R}$ является алгебраическим числом степени 2 и поэтому, согласно теореме

Лиувилля ([10], гл. 1 п.3), с некоторой постоянной $C > 0$ для любых натуральных m и k выполняется неравенство

$$\left|\frac{T}{R} - \frac{m}{k}\right| \geq \frac{C}{k^2}.$$

Отсюда получаем следующую оценку

$$|\delta_{mk}| = \pi k \left|\frac{T}{R} - \frac{m}{k}\right| \geq \frac{C}{k}. \quad (10)$$

Тогда из того что $|\delta_{mk}| < \frac{\pi}{2}$ и оценки (10) следует

$$\left|\sin\left(\frac{\pi k}{R}T\right)\right| = |\sin \delta_{mk}| \geq \frac{2C}{\pi k}. \quad (11)$$

Следовательно, оценку (7) можно переписать следующим образом

$$|u_{tt}(r, t)| \leq C \sum_{k=1}^{\infty} \frac{k^3 |\psi_k|}{|\sin(\lambda_k T)|} \leq C_1 \sum_{k=1}^{\infty} k^4 |\psi_k| \leq C_1 \sqrt{\sum_{k=1}^{\infty} k^{10} |\psi_k|^2} \cdot \sqrt{\frac{1}{k^2}}.$$

В силу условий наложенных на функцию $\psi(r)$, ряд в последнем неравенстве сходится/

Теорема доказана.

Регуляризация и сходимость к решению задачи.

Для любого $\alpha > 0$ и $p > 0$ введем функцию

$$u_{\alpha}(r, t) = \sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k}{R}T\right)}{\sin^2\left(\frac{\pi k}{R}T\right) + \alpha \lambda_k^p} \psi_k \sin\left(\frac{\pi k}{R}t\right) v_k(r). \quad (12)$$

Функция (12) обладает следующим свойством.

Теорема 2. Пусть $\alpha > 0$ и $s > 0$. Если $\psi \in H^{2s}(|x| < R)$, то функция $u_{\alpha}(r, t)$ принадлежит классу Соболева $W_2^{p+2s}(|x| < R)$.

Доказательство. Так как мы предполагаем, что функция (12) принадлежит классу

$W_2^{p+2s}(|x| < R) = H^{p+2s}(|x| < R)$, то необходимо доказать, что следующий ряд сходится

$$\sum_{k=1}^{\infty} k^{2p+4s} |u_{\alpha,k}|^2,$$

где $u_{\alpha,k}$ коэффициенты ряда Фурье функции (12) по системе функций (4).

Согласно оценке

$$\sin\left(\frac{\pi k}{R}T\right) + \frac{\alpha \lambda_k^p}{\sin\left(\frac{\pi k}{R}T\right)} \geq 2\sqrt{\alpha \lambda_k^p},$$

находим

$$\sum_{k=1}^{\infty} k^{2p+4s} |u_{\alpha,k}|^2 \leq \frac{C_2}{\alpha} \sum_{k=1}^{\infty} k^{2p+4s} \frac{|\psi_k|^2}{k^{2p}} = \frac{C_2}{\alpha} \|\psi\|_{H^{2s}(|x| < R)}^2 < \infty.$$

Теорема доказана.

Теперь исследуем, какой предел будет иметь функция (12) при $\alpha \rightarrow 0^+$. Имеет место следующая теорема.

Теорема 3. Пусть $\psi \in H^{2s}(|x| < R)$. Если $s > p + 4$ и число $\frac{T}{R}$ является алгебраическим числом степени 2, то $u_\alpha(r, t) \rightarrow u(r, t)$ при $\alpha \rightarrow 0^+$ равномерно по $|x| < R$ и $t \in [0, T]$.

Доказательство. Рассмотрим разность

$$u(r, t) - u_\alpha(r, t) = \alpha \sum_{k=1}^{\infty} \frac{\lambda_k^p}{\sin(\sqrt{\lambda_k}T)(\sin^2(\sqrt{\lambda_k}T) + \alpha\lambda_k^p)} \psi_k \sin(\sqrt{\lambda_k}t) v_k(r). \quad (13)$$

Оценим знаменатель ряда (13). Используя оценку (11), находим

$$|\sin(\sqrt{\lambda_k}T)|(\sin^2(\sqrt{\lambda_k}T) + \alpha\lambda_k^p) \geq 2\sqrt{\alpha\lambda_k^p} \sin^2(\sqrt{\lambda_k}T) \geq 2\sqrt{\alpha} \left(\frac{\pi k}{R}\right)^p \left(\frac{2C}{\pi k}\right)^2$$

или

$$|\sin(\sqrt{\lambda_k}T)|(\sin^2(\sqrt{\lambda_k}T) + \alpha\lambda_k^p) \geq C_3 \sqrt{\alpha} k^{p-2}.$$

Тогда из (13), получим

$$|u(r, t) - u_\alpha(r, t)| \leq C_4 \sqrt{\alpha} \sum_{k=1}^{\infty} \frac{\lambda_k^{p+\frac{1}{2}}}{k^{p-2}} |\psi_k| = C_5 \sqrt{\alpha} \sum_{k=1}^{\infty} k^{p+3} |\psi_k|.$$

Используя неравенство Коши-Буняковского, из последнего неравенства имеем

$$|u(r, t) - u_\alpha(r, t)| \leq C_5 \sqrt{\alpha} \sqrt{\sum_{k=1}^{\infty} k^{2p+8} |\psi_k|^2} \cdot \sqrt{\frac{1}{k^2}}.$$

Учитывая условия теоремы, получаем, что $|u(r, t) - u_\alpha(r, t)|$ будет стремиться к 0, при $\alpha \rightarrow 0^+$ равномерно по x и t .

Теорема доказана.

Регуляризация задачи с неточными данными.

Предположим, что вместо $\psi(r)$ задана функция $\psi_\delta(r)$ такая, что

$$\|\psi - \psi_\delta\|_{H^2(|x| < R)} < \delta. \quad (14)$$

Введем функцию

$$u_{\alpha,\delta}(r, t) = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\lambda_k}T)}{\sin^2(\sqrt{\lambda_k}T) + \alpha\lambda_k^p} \psi_{\delta,k} \sin(\sqrt{\lambda_k}t) v_k(r). \quad (15)$$

Если функция $\psi_\delta(r)$ удовлетворяет условиям теоремы 3, то функция (15) будет стремиться при $\alpha \rightarrow 0^+$ равномерно по $|x| < R$ и $t \in [0, T]$ к функции $u_\alpha(r, t)$, определенной формулой (13), в которой $\psi(r)$ заменена функцией $\psi_\delta(r)$.

Представляет интерес исследование вопроса о том, стремится ли функция (15) к решению (5) при α и δ стремящихся к нулю, а также установление характера этой сходимости. Для этого оценим разность

$$\begin{aligned} |u_{\alpha,\delta}(r, t) - u(r, t)| &= |u_{\alpha,\delta}(r, t) - u_\delta(r, t) + u_\delta(r, t) - u(r, t)| \leq \\ &\leq |u_{\alpha,\delta}(r, t) - u_\delta(r, t)| + |u_\delta(r, t) - u(r, t)| \end{aligned}$$

Из формул (5), (12) и (15), находим

$$\begin{aligned} u_{\alpha,\delta}(r, t) - u_\delta(r, t) &= \sum_{k=1}^{\infty} \left[\frac{\sin(\sqrt{\lambda_k}T)}{\sin^2(\sqrt{\lambda_k}T) + \alpha\lambda_k^p} - \frac{1}{\sin(\sqrt{\lambda_k}T)} \right] \psi_{\delta,k} \sin(\sqrt{\lambda_k}t) v_k(r), \\ u_\delta(r, t) - u(r, t) &= \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{R}t)}{\sin(\frac{\pi k}{R}T)} [\psi_{\delta,k} - \psi_k] v_k(r). \end{aligned}$$

Учитывая полученные ранее оценки и то, что число $\frac{T}{R}$ является алгебраическим числом степени 2, получим

$$|u_{\alpha,\delta}(r, t) - u(r, t)| \leq C(\sqrt{\alpha} \sum_{k=1}^{\infty} k^{p+3} |\psi_{\delta,k}| + \sum_{k=1}^{\infty} k^2 |\psi_{\delta,k} - \psi_k|). \quad (16)$$

Из оценки (16) можно получить следующее утверждение.

Теорема 4. Пусть $\psi_\delta \in H^{2s}(|x| < R)$ и удовлетворяет неравенству (14). Если $s > p + 4$ и число $\frac{T}{R}$ является алгебраическим числом степени 2, то $u_{\alpha,\delta}(r, t) \rightarrow u(r, t)$ при $\alpha \rightarrow 0^+$ и $\delta \rightarrow 0$ равномерно по $|x| < R$ и $t \in [0, T]$.

Доказательство. Используя неравенство Коши-Буняковского, мы можем переписать неравенство (16) следующим образом:

$$|u_{\alpha,\delta}(r,t) - u(r,t)| \leq C \left(\sqrt{\alpha} \sqrt{\sum_{k=1}^{\infty} k^{2p+8} |\psi_{\delta,k}|^2} \cdot \sqrt{\frac{1}{k^2}} + \sqrt{\sum_{k=1}^{\infty} k^6 |\psi_{\delta,k} - \psi_k|^2} \cdot \sqrt{\frac{1}{k^2}} \right).$$

Учитывая условия на заданную функцию, находим:

$$|u_{\alpha,\delta}(r,t) - u(r,t)| \leq C(\sqrt{\alpha} + \delta).$$

Из последнего неравенства следует утверждение теоремы.

Теорема доказана.

Заключение. В работе исследована задача Дирихле для волнового уравнения, описывающего малые радиальные колебания шара. Получено представление решения в виде спектрального разложения по системе собственных функций соответствующей краевой задачи.

Установлены условия разрешимости задачи, при которых рядовое представление решения сходится равномерно вместе с необходимыми производными. Показано, что при выполнении диофантовых условий на отношение T/R удаётся избежать явления малых знаменателей и обеспечить корректность построенного решения.

Предложен метод регуляризации, основанный на модификации спектрального разложения с использованием параметра регуляризации. Доказано, что регуляризованное решение обладает повышенной гладкостью и принадлежит соответствующим пространствам Соболева.

Исследована сходимость регуляризованного решения к точному решению при стремлении параметра регуляризации к нулю. Установлена равномерная сходимость по пространственной и временной переменным, при выполнении условий на гладкость исходных данных.

Рассмотрен случай неточных исходных данных. Получены оценки устойчивости решения и доказано, что регуляризованное решение сходится к точному при согласованном стремлении параметра регуляризации и уровня погрешности к нулю. Установлена оценка скорости сходимости вида $O(\alpha + \delta)$.

Полученные результаты могут быть использованы при исследовании некорректных краевых задач для гиперболических уравнений, а также при разработке устойчивых численных алгоритмов решения обратных и некорректных задач математической физики.

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КРАЕВАЯ ЗАДАЧА ДЛЯ НАГРУЖЕННОГО УРАВНЕНИЯ ПАРАБОЛО-
ГИПЕРБОЛИЧЕСКОГО ТИПА В ДВУСВЯЗНОЙ ОБЛАСТИ

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Аннотация. Настоящая работа посвящена постановке и исследованию локальной краевой задаче для нагруженного уравнения парабола-гиперболического типа в двухсвязной области, когда нагруженная часть содержит след или производную от искомой функции. Доказаны теоремы существования и единственности классического решения поставленных задач на основаны принципа экстремума и теории вольтеррских интегральных уравнений.

Ключевые слова: нагруженное уравнение, локальной краевой задачи, представление общего решение, принцип экстремума, интегральное уравнение Вольтерра.

MSC 2020: 35M10, 35M12, 35K15, 35L10, 35K10.

BOUNDARY VALUE PROBLEM FOR A LOADED PARABOLIC–HYPERBOLIC TYPE
EQUATION IN A DOUBLY CONNECTED DOMAIN

Abstract. This work is devoted to the formulation and investigation of a local boundary value problem for a loaded equation of parabolic–hyperbolic type in a doubly connected domain, where the loaded term contains the trace or a derivative of the unknown function. Theorems on the existence and uniqueness of a classical solution to the posed problems are proved based on the maximum principle and the theory of Volterra integral equations.

Keywords: loaded equation, local boundary value problem, representation of the general solution, maximum principle, Volterra integral equation.

MSC 2020: 35M10, 35M12, 35K15, 35L10, 35K10.

IKKI BOG‘LAMLI SOHADA PARABOLA-GIPERBOLIK TURDAGI YUKLANGAN
TENGLAMA UCHUN CHEGARAVIY MASALA

Annotatsiya. Mazkur ish ikki bog‘lamli sohada parabola-giperbolik turdagi yuklangan tenglama uchun lokal chegaraviy masalani qo‘yish va tadqiq etishga bag‘ishlangan. Bunda yuklangan qism izlanayotgan funksiyaning izi yoki hosilasini o‘z ichiga oladi. Qo‘yilgan masalalar uchun klassik yechimning mavjudligi va yagonaligi haqidagi teoremlar ekstremum prinsipi hamda Volterra integral tenglamalari nazariyasiga asoslanib isbotlangan.

Kalit so‘zlar: yuklangan tenglama, lokal chegaraviy masala, umumiy yechimni ifodalash, ekstremum prinsipi, Volterra integral tenglamasi.

MSC 2020: 35M10, 35M12, 35K15, 35L10, 35K10.

Введение. Начиная с пятидесятих годов прошлого века, исследователи интенсивно изучают локальные и нелокальные задачи для уравнений в частных производных в односвязных областях. В настоящее время круг таких задач расширяется в различных направлениях. Среди них особое место занимают задачи со смещением. Это объясняется тем, что такие задачи охватывают различные корректные локальные краевые задачи, и к ним сводятся многие проблемы, например биологической синергетики, генетики, иммунологии, трансзвуковой газовой динамики и тепловой физики.

Одним из важнейших классов ненагруженных уравнений с частными производными являются уравнения второго порядка парабола-гиперболического и эллиптико - гиперболического типов. Аналог задачи Трикоми в двухсвязной области для уравнения смешанного типа с одной линией вырождения впервые поставлен и изучен в работе А.В.Бицадзе [1, стр.30], а для уравнения Лаврентьева–Бицадзе — в работе М. С. Салахитдинова и А. К. Уринова[2].

Краевые задачи в классических областях для нагруженного уравнения гиперболического типа изучены в работах А.М.Нахушева[3],

В.М.Казиева[4], А.Х.Аттаева[5], а М.Т.Дженалиев[6] и М.Х.Шханков[7] исследовали локальные и нелокальные задачи для нагруженного уравнения параболического типа.

А.В.Бородин[8] изучал задача Дирихле для нагруженного уравнения эллиптического типа. В работах К.У.Хубиева[9], Б.Исломова и Д.М.Курызова[10], М.И.Рамазанова [11], изучены аналоги задачи Трикоми и Геллерстедта для нагруженных уравнений гипербола-параболического типа, а задача Дирихле для нагруженного уравнения с оператором Лаврентьева– Бицадзе в прямоугольной области исследованы в работах [12-14].

Локальные задачи в односвязных областях для смешанных нагруженных уравнений, когда нагруженные части содержит след или производную от искомой функции мало изучены. Отметим работы [15-21]. Это связано, прежде всего, с отсутствием представления общего решения для таких уравнений; с другой стороны, такие задачи сводятся к малоизученным интегральным уравнениям со сдвигом.

В настоящее время круг таких задач расширяется в различных специальных областях. Однако краевые задачи для нагруженных уравнений смешанного типа в двусвязных областях до сих пор недостаточно исследованы. Отметим работы [22-23]. В работах [24-25] разработать новый метод к построению представление общего решения нагруженного парабола-гиперболического уравнения в форме, удобной для дальнейших исследований различных краевых задач. Также показано, что на корректность таких краевых задач существенно влияет нагруженная часть рассматриваемого уравнения.

Исходя из этого, настоящая работа посвящена постановке и изучению локальной краевой задачи для нагруженного уравнения парабола-гиперболического типа в двусвязной области.

Постановка задачи TS_μ

Пусть Ω_0 – область, ограниченная отрезками $A_j B_j$, $A_j M_j \cup M_j K_j$, $B_j N_j \cup N_j P_j$, $K_j P_j \cup P_j C_4$, $N_j C_3$ прямых $y=0$, $x=(-1)^{j-1}$, $x=(-1)^{j-1}q$, $y=1$, $y=h$ при $y>0$. Δ_j – область, ограниченная отрезками $A_j B_j$ оси Ox и $C_1 C_2$ оси Oy а при $y<0$ характеристиками $A_j C_1 : x - (-1)^{j-1} y = (-1)^{j-1}$, $B_j C_2 : x - (-1)^{j-1} y = (-1)^{j-1} q$ ($j=1,2$) уравнения

$$0 = \begin{cases} u_{xx} - u_y - \mu_j u_x(x, 0), & (x, y) \in \Omega_{0j}, \\ u_{xx} - u_y - p_j u(0, y), & (x, y) \in \Omega_{0j+2}, \\ u_{xx} - u_{yy} - \mu_{j+2} u(x+y, 0), & (x, y) \in \Delta_j \end{cases} \quad (1)$$

выходящими из точек $A_j((-1)^{j-1}; 0)$ и $B_j((-1)^{j-1}q; 0)$, пересекающимися в точках

$C_1(0; -1)$ и $C_2(0; -q)$, здесь $\Omega_0 = \Omega_{01} \cup \Omega_{02} \cup \Omega_{03} \cup \Omega_{04}$, $\Omega_{0j} = A_j K_j P_j B_j$, $\Omega_{0j+2} = N_j C_3 C_4 P_j$, $\Delta_j = A_j B_j D_j \cup B_j D_j E_j C_2 \cup C_2 E_j C_1$ ($j=1,2$). В уравнение (1) h, q, μ_j, p_j ($j=1,2$) – заданные действительные числа, причем

$$h > 0, 0 < q < 1, (-1)^{j-1} \mu_j \leq 0, \mu_{j+2} \leq 0, p_j \geq 0, (j=1,2). \quad (2)$$

Введем обозначения: $I_1 = \{(x, y) : x=0, -1 < y < -q\}$, $I_2 = \{(x, y) : x=0, h < y < 1\}$,

$$I_3 = \{(x, y) : x=q, h < y < 1\}, I_4 = \{(x, y) : x=-q, h < y < 1\},$$

$$J_1 = \{(x, y) : q < x < 1, y=0\}, J_2 = \{(x, y) : -1 < x < -q, y=0\},$$

$$D_j, E_j \in A_j C_1, B_j D_j : x + (-1)^{j-1} y = (-1)^{j-1} q,$$

$$C_2 E_j : x + (-1)^{j-1} y = (-1)^j q, \quad \left. \begin{matrix} A_j D_j \\ A_j E_j \end{matrix} \right\} : x - (-1)^{j-1} y = (-1)^{j-1}, (j=1,2).$$

Через Ω_j и Ω_{j+2} , Ω_5 соответственно обозначим характеристические треугольники $A_j B_j D_j$ и четырехугольники $B_j C_2 E_j D_j$, $C_2 E_1 C_1 E_2$, .

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup B_1 D_1 \cup C_2 E_1 \cup C_2 E_2 \cup B_2 D_2,$$

$$\Omega_{5j} = \Omega_5 \cap \{(-1)^j x < 0, y < 0\} = C_2 E_j C_1,$$

$$\Delta_j = \Omega_j \cup \Omega_{j+2} \cup \Omega_{5j}, \Delta = \Omega_0 \cup \Delta_1 \cup \Delta_2 \cup I_1 \cup I_2 \cup J_1 \cup J_2, \quad \Delta_j^* = \Omega_j \cup \Omega_{0j}, \quad (j=1,2).$$

Задача TS_μ . Найти функцию $u(x, y)$, обладающую следующими свойствами:

1) $u(x, y) \in C(\bar{\Delta}) \cap C^1(\Delta)$; 2) $u(x, y) \in C_{x,y}^{2,1}(\Omega_{0j} \cup \Omega_{0j+2} \cup K_j C_4) \cap C_{x,y}^{2,2}(\Delta_j)$ и удовлетворяет уравнению (1) в областях Ω_{0j} , Ω_{0j+2} и Δ_j ($j=1,2$); 3)

$u(x, y)$ удовлетворяет краевым условиям:

$$u(x, y)|_{A_j K_j} = \varphi_j(y), \quad 0 \leq y \leq 1, \quad u(x, y)|_{B_j N_j} = g_j(y), \quad 0 \leq y \leq h, \quad (3)$$

$$u(x, y)|_{N_j C_3} = f_j(x), \quad 0 \leq (-1)^{j-1} x \leq q, \quad (4)$$

$$u(x, y)|_{A_j D_j} = \psi_j(x), \quad \frac{1+q}{2} \leq ((-1)^{j-1} x) \leq 1, \quad (5)$$

$$u(x, y)|_{B_j C_2} = \psi_{j+2}(x), \quad 0 \leq ((-1)^{j-1} x) \leq q, \quad (j=1,2); \quad (6)$$

4) на линиях I_j , J_j и I_{j+2} ($j=1,2$) выполняются условиям склеивания

$$u_y(x, +0) = u_y(x, -0), \quad (x, 0) \in J_j, \quad (7)$$

$$u_x(-0, y) = u_x(+0, y), \quad (0, y) \in I_2, \quad (8)$$

$$u_x((-1)^{j-1} q + 0, y) = u_x((-1)^{j-1} q - 0, y), \quad ((-1)^{j-1} q, y) \in I_{j+2}, \quad (9)$$

где $\varphi_j(y)$, $g_j(y)$, $f_j(x)$, $\psi_j(x)$, $\psi_{j+2}(x)$ ($j=1,2$) – заданные функции, причем

$$\psi_j((-1)^{j-1}) = \varphi_j(0) = 0, \quad g_j(h) = f_j((-1)^{j-1} q) = 0, \quad (10)$$

$$\psi_{j+2}((-1)^{j-1} q) = g_j(0) = 0, \quad \psi_3(0) = \psi_4(0), \quad f_1(0) = f_2(0) = 0,$$

$$\varphi_j(y) \in C[0,1] \cap C^1(0,1), \quad g_j(y) \in C[0,h] \cap C^1(0,h), \quad (11)$$

$$\psi_j(x) \in C^1\left(\frac{1+q}{2} \leq ((-1)^{j-1} x) \leq 1\right) \cap C^2\left(\frac{1+q}{2} < ((-1)^{j-1} x) < 1\right), \quad (12)$$

$$f_j(x) \in C\left(0 \leq ((-1)^{j-1} x) \leq q\right) \cap C^2\left(0 < ((-1)^{j-1} x) < q\right), \quad (13)$$

$$\psi_{j+2}(x) \in C^1\left(0 \leq ((-1)^{j-1} x) \leq q\right) \cap C^2\left(0 < ((-1)^{j-1} x) < q\right). \quad (14)$$

Исследование задачи TS_μ

Для исследование задачи TS_μ важную роль играют следующие задачи теоремы..

Задача $TS_{j\mu}$ ($j=1,2$). Найти функцию $u(x, y)$, обладающую следующими свойствами:

1) $u \in C(\bar{\Delta}_j^*) \cap C^1(\Delta_j^* \cup \bar{J}_j)$; 2) $u \in C_{x,y}^{2,1}(\Omega_{0j} \cup K_j P_j) \cap C_{x,y}^{2,2}(\Omega_j)$ и удовлетворяет

уравнению (1) в областях $\Omega_{0j} \cup K_j P_j$ и Ω_j ; 3) $u(x, y)$ удовлетворяет краевым условиям (3), (5), (7) и

$$u|_{N_j P_j} = \tau_{j+2}(y), \quad (0, y) \in \bar{I}_{j+2}, \quad (15)$$

где $\tau_{j+2}(y)$ – заданная функция, подлежащая определению, причем

$$\tau_{j+2}(y) \in C[h, 1] \cap C^1(h, 1), \quad (j=1, 2). \quad (16)$$

Теорема 1. Если выполнены условия (2), (10)-(12), (16), то в области Δ_j^* регулярное решение задачи $TS_{j\mu}$ ($j=1, 2$) существует и единственно.

Теорема 1 доказывается методом интегральных уравнений. Согласно этому задача $TS_{j\mu}$ ($j=1, 2$) эквивалентным образом сводится к интегральным уравнениям Вольтерра второго рода. Однозначная разрешимость задачи следует из однозначной разрешимости интегральных уравнений Вольтерра[26].

Задача B_j ($j=1, 2$). Найти функцию $u(x, y)$, обладающую следующими свойствами: 1) $u \in C(\bar{\Omega}_{0j}) \cap C^1(\Omega_{0j})$; 2) $u \in C_{x,y}^{2,1}(\Omega_{0j} \cup K_j P_j)$ и удовлетворяет уравнению (1) в областях $\Omega_{0j} \cup K_j P_j$ ($j=1, 2$); 3) $u(x, y)$ удовлетворяет краевым условиям (3), (15) и

$$u(x, +0) = \tau_j(x), \quad (x, 0) \in \bar{J}_j, \quad (17)$$

где $\tau_j(x)$ – заданная функция, подлежащая определению, причем

$$\tau_j(x) \in C^1(\bar{J}_j) \cap C^3(J_j). \quad (18)$$

Теорема 2. Если выполнены условия (2), (11), (13), (18) и

$$\tau_j((-1)^{j+1}) = \varphi_j(0) = 0, \quad g_j(h) = f_j((-1)^{j-1}q) = \tau_{j+2}(h), \quad \tau_j((-1)^{j-1}q) = g_j(0) = 0,$$

то в области Ω_{0j} регулярное решение первой краевой задачи B_j ($j=1, 2$) существует и единственно.

Задача B_{j+2} ($j=1, 2$). Найти функцию $u(x, y)$, обладающую следующими свойствами: 1) $u \in C(\bar{\Omega}_{0j+2}) \cap C^1(\Omega_{0j+2})$; 2) $u \in C_{x,y}^{2,1}(\Omega_{0j} \cup P_j C_4)$ и удовлетворяет уравнению (1) в областях $\Omega_{0j} \cup P_j C_4$ ($j=1, 2$); 3) $u(x, y)$ удовлетворяет краевым условиям (4), (15) и $u|_{C_3 C_4} = \tau_5(y)$, $(0, y) \in \bar{I}_2$, где $\tau_5(y)$ – заданная функция, причем $\tau_5(h) = f_j(0) = 0$,

$$\tau_5(y) \in C(\bar{I}_2) \cap C^1(I_2). \quad (19)$$

Теорема 3. Если выполнены условия (2), (13), (19), $f_j((-1)^{j-1}q) = \tau_{j+2}(h) = 0$, и $\tau_5(h) = f_j(0) = 0$, то в области Ω_{0j+2} регулярное решение первой краевой задачи B_{j+2} ($j=1, 2$) существует и единственно.

Используя, решение первой краевой задачи для уравнения (1) в областях Ω_{0j} и Ω_{0j+2} ($j=1, 2$) доказываются теоремы 2 и 3.

Задача B_5 . Найти функцию $u(x, y)$, обладающую следующими свойствами: 1) $u \in C(\bar{\Omega}_0) \cap C^1(\Omega_0)$; 2) $u \in C_{x,y}^{2,1}(\Omega_{0j} \cup \Omega_{0j+2} \cup K_j P_j \cup P_j C_4)$ и удовлетворяет уравнению

(1) в областях $\Omega_{0j} \cup \Omega_{0j+2} \cup K_j P_j \cup P_j C_4 (j=1,2)$; 3) $u(x, y)$ удовлетворяет краевым условиям (3), (4), (8), (9) и (17).

Теорема 4. Если выполнены условия (2), (11), (13), (18) и $f_1(0) = f_2(0) = 0$,

$$\tau_j((-1)^{j+1}) = \varphi_j(0) = 0, \quad g_j(h) = f_j((-1)^{j-1}q) = 0, \quad \tau_j((-1)^{j-1}q) = g_j(0) = 0, \quad (j=1,2),$$

то в области Ω_0 регулярное решение задачи B_5 существует и единственно.

Теорема 4 доказывается методом интегральных уравнений. Согласно этому задача B_5 эквивалентным образом сводится к системе интегральных уравнений Вольтерра второго рода. Однозначная разрешимость задачи следует из однозначной разрешимости интегрального уравнения Вольтерра.

Задача $\Gamma_j (j=1,2)$. Найти функцию $u(x, y)$ со следующими свойствами: 1) $u \in C(\bar{\Omega}_{j+2}) \cap C^1(\Omega_{j+2})$; 2) $u(x, y)$ - дважды непрерывно дифференцируемое решение уравнения (1) в области Ω_{j+2} ; 3) $u(x, y)$ удовлетворяет краевым условиям (6) и $u(x, y)|_{B_j D_j} = w_j(x)$, $q \leq (-1)^{j-1}x \leq \frac{1+q}{2}$, где $w_j(x) (j=1,2)$ - заданные функции, причем $w_j((-1)^{j-1}q) = \psi_j((-1)^{j-1}q)$,

$$w_j(x) \in C\left(q \leq (-1)^{j-1}x \leq \frac{1+q}{2}\right) \cap C^2\left(q < (-1)^{j-1}x < \frac{1+q}{2}\right). \quad (20)$$

Теорема 5. Если выполнены условия (2), (14), (20), то в области Ω_{j+2} решение задачи Γ_j существует и единственно.

Задача Γ_3 . Найти функцию $u(x, y)$ со следующими свойствами: 1) $u \in C(\bar{\Omega}_5) \cap C^1(\Omega_5)$, причем функция $u_x(0, y)$ может обращаться в бесконечность порядка меньше единицы на концах интервала I_1 ; 2) $u(x, y)$ - дважды непрерывно дифференцируемое решение уравнения (1) в области $\Omega_{5j} (j=1,2)$; 3) $u(x, y)$ удовлетворяет краевым условиям $u(x, y)|_{C_2 E_j} = h_j(y)$, $-\frac{1+q}{2} \leq y \leq -q$, где $h_j(y) (j=1,2)$ - заданные функции, причем $h_1(-q) = h_2(-q)$,

$$h_j(y) \in C\left[-\frac{1+q}{2}, -q\right] \cap C^2\left(-\frac{1+q}{2}, -q\right). \quad (21)$$

Теорема 6. Если выполнены условия (2), (21), то в области Ω_5 решение задачи Γ_3 существует и единственно.

Используя, общие решение уравнения (1) в областях Ω_{j+2} и $\Omega_{5j} (j=1,2)$ доказываются теоремы 5 и 6.

Следовательно, решение задачи TS_μ можно восстановить в области Ω_0 как решение первой краевой задачи для уравнения (1), а в области $\Omega_j (\Omega_{j+2} \cup \Omega_5)$ как решение задачи Коши(Гурса) для уравнения (1). Этим завершается исследование задачи TS_μ для уравнения (1) в области Δ .

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О ПОЛУПЕРИОДИЧЕСКОЙ КРАЕВОЙ ЗАДАЧЕ В НЕОГРАНИЧЕННОМ ПОЛЕ ДЛЯ
УРАВНЕНИЯ СМЕЩАННОГО ТИПА

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Аннотация. В статье излагается о единственности и существование обобщенного решения полупериодической краевой задачи для трехмерного уравнения Трикоми в виде неограниченного параллелепипеда. Для доказательства теоремы о единственности и существования решения задачи используется преобразование Фурье, методы "ε – регуляризации" и методов априорных оценок.

Ключевые слова: трехмерное уравнения Трикоми, периодическая краевая задача, преобразование Фурье, методы "ε – регуляризации", априорных оценок.

APALASH TIPDAGI TENGLAMA UCHUN CHEGARALANMAGAN SOHADA
YARIMDAVRIY CHEGARAVIY MASALA HAQIDA

Annotatsiya. Ushbu maqolada chegaralanmagan parallelepiped ko'rinishidagi sohada uch o'lchovli Triкоми tenglamasi uchun davriy tipdagi chegaraviy masalani umumlashgan yechiminining yagonaligi va mavjudligi o'rganilgan. Yagonalik va mavjudlik teoremlarini isbotlash uchun Furye almashtirishi, "ε - regularizatsiya" va aprior baholar usullaridan foydalanamiz.

Kalit so'zlar: uch o'lchovli Triкоми tenglamasi, yarimdavriy tipdagi chegaraviy masala, "ε - regularizatsiyasi", aprior baholar usullari, Furye almashtirishi.

ON A SEMI-PERIODIC BOUNDARY VALUE PROBLEM IN AN UNBOUNDED FIELD FOR
AN EQUATION OF OFFSET TYPE

Abstract. In this paper, we investigate uniqueness, existence and smoothness of a generalized solution to the periodic boundary value problem for three-dimensional Triкоми equation in unbounded parallelepiped. In order to prove the theorems of uniqueness, existence and smoothness of a generalized solution, we mainly use the Fourier transform, the methods of "ε – regularization" and the priori estimates as well.

Keywords: three-dimensional Triкоми equation, periodic boundary value problem, "ε – regularization" method, a priori estimates, Fourier transform.

Введение и постановки задачи. Как известно, в работе А.В.Бицадзе показано, что задача Дирихле для уравнения смешанного типа некорректна [1]. Естественно возникает вопрос: нельзя ли заменить условия задачи Дирихле другими условиями, охватывающими всю границу, которые обеспечивают корректность задачи? Впервые такие краевые задачи (нелокальные краевые задачи) для уравнения смешанного типа были предложены и изучены в работе Ф.И.Франкля [2]. Как близкие по постановке к изучаемым, задачи для уравнения смешанного типа первого рода исследована в ограниченных областях в работах [3,4].

В данной работе с использованием результатов работ [3,4,5] изучаются однозначная разрешимость и гладкость обобщенного решения одной полупериодической краевой задаче в неограниченном параллелепипеде.

В области

$$G = (-1, 1) \times (0, T) \times R = Q \times R = \{(x, t, z); x \in (-1, 1), 0 < t < T < +\infty, z \in R.\}$$

рассмотрим уравнения Трикоми:

$$Lu = xu_{tt} - \Delta u + a(x, t)u_t + c(x, t)u = f(x, t, z), \quad (1)$$

где $\Delta u = u_{xx} + u_{zz}$ - оператор Лапласа.

Пусть все коэффициенты уравнения (1) достаточно гладкие функции в области Q . В дальнейшем для решения поставленных задач нам необходимо ввести определений несколько функциональных пространств и обозначение.

Обозначим через $\hat{u}(x, t, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} u(x, t, z) e^{-i\lambda z} dz$

преобразование Фурье, функции $u(x, t, z)$, а через

$u(x, t, z) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{u}(x, t, \lambda) e^{i\lambda z} dz$ обозначим обратное преобразование Фурье. Теперь с

помощью преобразование Фурье определим пространство $W_2^{l,s}(G)$ с нормой

$$\|u\|_{W_2^{l,s}(G)}^2 = (2\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} (1 + |\lambda|^2)^s \cdot \|\hat{u}(x, t, \lambda)\|_{W_2^l(Q)}^2 d\lambda, \quad (A)$$

где s, l — любые конечные положительные целые числа, а норма в пространстве Соболева $W_2^l(Q)$, определяется следующим образом

$$\|g\|_{W_2^l(Q)}^2 = \sum_{|\alpha| \leq l} \int_Q |D^\alpha g|^2 dx dt, \quad \alpha \text{ — это мультииндекс, } D^\alpha \text{ — есть обобщённая}$$

производная по переменным x и t . Очевидно, что пространство $W_2^{l,s}(G)$ с нормой (A) является банаховым пространством [6,7,10].

Полупериодическая краевая задача. Найти обобщённое решение $u(x, t, z)$ уравнения (1) из пространства $W_2^2(G)$, удовлетворяющее следующим краевым условиям

$$D_t^p u|_{t=0} = D_t^p u|_{t=T}, \quad (2)$$

$$u|_{x=-1} = u|_{x=1} = 0 \quad (3)$$

при $p = 0, 1$, где $D_t^p u = \frac{\partial^p u}{\partial t^p}$, $D_t^0 u = u$, отличное от нуля, величина которого будет

уточнена ниже.

Определение 1. Обобщённым решением задачи (1)-(3) будем называть функцию $u(x, t, z) \in W_2^2(G)$, удовлетворяющую уравнению (1) почти всюду в области G , с условиями (2)-(3).

Теорема 1. (Основной результат). Пусть выполнены следующие условия для коэффициентов уравнения (1); $2a(x, t) + \mu x \geq \delta_1 > 0$, $\mu c(x, t) - c_t(x, t) \geq \delta_2 > 0$ для всех $(x, t) \in \overline{Q}$, где $\mu - \text{const} > 0$, $a(x, 0) = a(x, T)$, $c(x, 0) = c(x, T)$ для всех $x \in [-1, 1]$. Тогда для любой функции $f \in W_2^{1,3}(G)$ такой, что $f(x, 0, z) = f(x, T, z)$, существует единственное обобщенное решение задачи (1)-(3) из пространства $W_2^{2,3}(G)$ и для решения задачи (1)-(3) справедливы следующие оценки:

$$I) \quad \|u\|_{W_2^{1,3}(G)}^2 \leq c_1 \|f\|_{W_2^{0,3}(G)}^2$$

$$II) \quad \|u\|_{W_2^{2,3}(G)}^2 \leq c_2 \|f\|_{W_2^{1,3}(G)}^2,$$

в дальнейшем через C_i — обозначим положительное, вообще говоря, разные постоянные числа, отличные от нуля.

Однозначное разрешимость задачи (1)-(3) докажем с помощью преобразованием Фурье. Применяя для задачи (1)-(3) преобразование Фурье получим в области $Q = (-1, 1) \times (0, T)$ следующую задачу

$$L\hat{u} = x\hat{u}_{tt} - \hat{u}_{xx} + a(x, t)\hat{u}_t + (c(x, t) + \lambda^2)\hat{u} = \hat{f}(x, t, \lambda), \quad (4)$$

$$D_t^p \hat{u}|_{t=0} = D_t^p \hat{u}|_{t=T}; p = 0, 1 \quad (5)$$

$$\hat{u}|_{x=-1} = \hat{u}|_{x=1}; \quad (6)$$

$$\text{где, } \lambda \in R = (-\infty, \infty), \quad \hat{f}(x, t, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(x, t, z) e^{-i\lambda z} dz$$

Преобразование Фурье по переменной z , функции $f(x, t, z)$.

Однозначная разрешимость и гладкость задачи (4)-(6) изучена в работах [3,4,5]. Коротко приведем эти результаты.

Единственность решения задачи (4)-(6).

Теорема 2. Пусть выполнены следующие условия для коэффициентов уравнения (4); $2a(x, t) - \mu x > \delta_1 > 0$, $\mu c(x, t) - c_t(x, t) > \delta_2 > 0$, где $\mu - \text{const} > 0$, для всех $(x, t) \in \bar{Q}$, $a(x, 0) = a(x, T)$, $c(x, 0) = c(x, T)$, для всех $x \in [-1, 1]$. Тогда, если для любой функции $\hat{f}(x, t, \lambda) \in L_2(Q)$ существует обобщенное решение задачи (4)-(6) из пространства $W_2^2(G)$, то оно единственно.

Доказательство. Докажем единственность решения задачи (4)-(6) с помощью метода интеграла энергии. Пусть существует решение задачи (4)-(6) из $W_2^2(Q)$. Рассмотрим тождество

$$2(L\hat{u}, \hat{u}_t + \mu\hat{u})_0 = 2(f, \hat{u}_t + \mu\hat{u})_0, \quad (7)$$

где $\mu = \text{const} > 0$. В силу условий теоремы 2, интегрируя по частям тождество (7), используя неравенства Коши с σ [7], легко получить следующую необходимую первую оценку

$$\|\hat{u}\|_{W_2^1(Q)}^2 \leq c_1 \|\hat{f}\|_{L_2(Q)}^2. \quad (8)$$

из которой следует единственность решения задачи (4)-(6) из $W_2^2(Q)$. **Теорема 2 доказана.**

Уравнение третьего порядка с малым параметром. Разрешимость задачи (4)-(6) докажем методом "ε-регуляризации", а именно, в области $Q = (-1, 1) \times (0, T)$ рассмотрим семейство уравнений третьего порядка с малым параметром

$$L_\varepsilon \hat{u}_\varepsilon = -\varepsilon \frac{\partial^3 \hat{u}_\varepsilon}{\partial t^3} + L\hat{u}_\varepsilon = \hat{f}(x, t, \lambda) \quad (9)$$

и с периодическими краевыми условиями

$$D_t^q \hat{u}_\varepsilon|_{t=0} = D_t^q \hat{u}_\varepsilon|_{t=T}; q = 0, 1, 2, \quad (10)$$

$$\hat{u}|_{x=-1} = \hat{u}|_{x=1} = 0 \quad (11)$$

где ε — малое положительное число, $D_z^q w = \frac{\partial^q w}{\partial z^q}$, $q = 1, 2$; $D_z^0 w = w$.

Ниже используем системы уравнений третьего порядка с малым параметром (9) в качестве «ε-регуляризующего» уравнения для уравнения Трикоми (4) [4,5,8,11].

Определим пространства функции $W(Q) = \{\hat{u}_\varepsilon; \hat{u}_\varepsilon \in W_2^2(Q), \hat{u}_{\varepsilon tt} \in L_2(Q)\}$ удовлетворяющих соответствующим условиям (11),(12) с конечной нормой

$$\|\hat{u}_\varepsilon\|_W^2 = \varepsilon \|\hat{u}_{\varepsilon ttt}\|_0^2 + \|\hat{u}_\varepsilon\|_2^2. \quad (B)$$

Очевидно, что пространство $W(Q)$ с нормой (B) является банаховым пространством [6,7].

Определение 2. Обобщенным решением задачи (9)-(11) будем называть функцию $\{\hat{u}_\varepsilon(x, t, \lambda)\} \in W(Q)$, удовлетворяющую уравнению (9) почти всюду в области Q , с условиями (10),(11).

Теорема 3. Пусть выполнены следующие условия для коэффициентов уравнения (9); кроме того, пусть $2a(x, t) - \mu x \geq \delta_1 > 0$, $\mu c(x, t) - c_t(x, t) \geq \delta_2 > 0$, где $\mu - const > 0$, для всех $(x, t) \in \bar{Q}$, $a(x, 0) = a(x, T)$, $c(x, 0) = c(x, T)$, для всех $x \in [-1, 1]$. Тогда, если для любой функции $\hat{f}(x, t, \lambda) \in L_2(Q)$ такой, что $f(x, 0, z) = f(x, T, z)$ существует обобщенное решение задачи (9)-(10) из пространства $W(Q)$ и для нее справедливы следующие оценки

$$III) \quad \varepsilon \|\hat{u}_{\varepsilon tt}\|_0^2 + \|\hat{u}_\varepsilon\|_1^2 \leq c_1 \|\hat{f}\|_0^2,$$

$$IV) \quad \varepsilon \|\hat{u}_{\varepsilon ttt}\|_0^2 + \|\hat{u}_\varepsilon\|_2^2 \leq c_2 \|\hat{f}\|_1^2,$$

Доказательство. Доказательство теоремы 3 осуществляется поэтапно, с использованием метода Галеркина с получением соответствующих априорных оценок [7]. Сначала докажем *III)* – третью оценку.

Рассмотрим тождество:

$$2 \int_Q L_\varepsilon \hat{u}_\varepsilon \cdot (\hat{u}_{\varepsilon t} + \mu \hat{u}_\varepsilon) dx dt = 2 \int_Q \hat{f} \cdot (2\hat{u}_{\varepsilon t} + \mu \hat{u}_\varepsilon) dx dt. \quad (12)$$

Интегрируя по частям тождество (12), учитывая условие теоремы 3 нетрудно получить *III)* – третью априорную оценку, аналогичную оценке (8), откуда следует единственность обобщенного решения задачи (9)-(11).

Теперь докажем справедливость *IV)* – четвертую оценку

Для этого рассмотрим тождество:

$$\int_Q e^{-\mu t} \cdot L_\varepsilon \hat{u}_\varepsilon \cdot P\hat{u}_\varepsilon dx dt = \int_Q e^{-\mu t} \cdot \hat{f} \cdot P\hat{u}_\varepsilon dx dt \quad (13)$$

$$\text{где } P\hat{u}_\varepsilon = (-2\hat{u}_{\varepsilon ttt} + \mu \hat{u}_{\varepsilon tt} - \mu \hat{u}_{\varepsilon xx} + \mu \hat{u}_{\varepsilon t}).$$

Интегрируя по частям (13), с учетом условий теоремы 3 и краевых условий (10), (11), получим необходимую четвертую оценку:

$$\varepsilon \|\hat{u}_{\varepsilon ttt}\|_0^2 + \|\hat{u}_\varepsilon\|_2^2 \leq c_2 \|\hat{f}\|_1^2 \quad (14)$$

Из доказанных оценок получим однозначную разрешимость задачи (9)-(11) из пространства $W(Q)$. **Теорема 3 доказана.**

Перейдем к доказательству разрешимости задачи (4) - (6).

Теорема 4. Пусть выполнены все условия теоремы 2,3. Тогда обобщенное решение задачи (4) - (6) существует и единственно в $W_2^2(Q)$.

Доказательство. Единственность решения задачи (4)-(6) в пространстве $W_2^2(Q)$ доказано в теореме 2. Теперь докажем существование решения задачи (4)-(6) в $W(Q)$. Для этого рассмотрим в области Q уравнение (9) и краевые условия (10),(11) при $\varepsilon > 0$. Так как выполнены все условия теоремы 3, то существует единственное обобщенное решение задачи (9)-(11) в $W(Q)$, при $\varepsilon > 0$ и для нее справедлива третья и четвертая оценка. Отсюда следует, по известной теореме о компактности [7], что из множества функций $\{\hat{u}_\varepsilon(x, t, \lambda)\}, \varepsilon > 0$ можно извлечь слабо

сходящуюся подпоследовательность функций, такую, что $\{\hat{u}_{\varepsilon_i}(x, t, \lambda)\} \rightarrow \hat{u}(x, t, \lambda)$ при $\varepsilon_i \rightarrow 0$ в $W(Q)$. Покажем, что предельная функция $\hat{u}(x, t, \lambda)$ удовлетворяет уравнению $L\hat{u} = \hat{f}$ (уравнению (4)) почти всюду в $W_2^2(Q)$. В самом деле, так как подпоследовательность $\{\hat{u}_{\varepsilon_i}(x, t, \lambda)\}$ слабо сходится в $W(Q)$, а подпоследовательность $\{\sqrt{\varepsilon_i} \hat{u}_{\varepsilon_i ttt}(x, t, \lambda)\}$ равномерно ограничена в $L_2(Q)$ и оператор L линейный, то имеем

$$L\hat{u} - \hat{f} = L\hat{u} - L\hat{u}_{\varepsilon_i} + \varepsilon_i \frac{\partial^3 \hat{u}_{\varepsilon_i}}{\partial t^3} = L(\hat{u} - \hat{u}_{\varepsilon_i}) + \varepsilon_i \frac{\partial^3 \hat{u}_{\varepsilon_i}}{\partial t^3}. \quad (15)$$

Из равенства (15), переходя к пределу при $\varepsilon_i \rightarrow 0$, получим единственное обобщенное решение задачи (4)-(6) из пространства Соболева $W_2^2(Q)$ [4,5,8,9,10,11]. Таким образом, **Теорема 4 доказана.**

Существование решения задачи (1)-(3). Теперь перейдем к доказательству теоремы 1, об однозначной разрешимости обобщенного решения задачи (1)-(3) из пространства $W_2^{2,3}(G)$.

Доказательство теоремы-1. В теореме 2 для решения задачи (4)-(6), доказана справедливость оценки (8), то есть следующее

$$\|\hat{u}\|_{W_2^1(Q)}^2 \leq c_1 \|\hat{f}\|_{L_2(Q)}^2. \quad (8)$$

$u_z \in L_2(G)$ Умножая неравенство (8) на $(2\pi)^{-1/2} \cdot (1 + |\lambda|^2)^3$ и интегрируя по λ от $-\infty$ до $+\infty$ получим

$$\begin{aligned} \|u\|_{W_2^{1,3}(G)}^2 &= (2\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} (1 + |\lambda|^2)^3 \cdot \|\hat{u}\|_{W_2^1(Q)}^2 d\lambda \leq \\ &\leq (2\pi)^{-1/2} \cdot c_1 \cdot \int_{-\infty}^{+\infty} (1 + |\lambda|^2)^3 \cdot \|\hat{f}\|_{L_2(Q)}^2 d\lambda = c_1 \|f\|_{W_2^{0,3}(G)}^2 \end{aligned} \quad (16)$$

из которой следует выполнение первой априорной оценки теоремы 1 и единственность обобщенного решения задачи (1)-(3) из $W_2^{2,3}(G)$.

Так, же используя условие теоремы-3,4 с предельным переходом при $\varepsilon \rightarrow 0$, в четвертой оценке нетрудно получить для решения задачи (1)-(3) выполнение следующие оценки

$$\|\hat{u}\|_{W_2^2(Q)}^2 \leq c_2 \|\hat{f}\|_{W_2^1(Q)}^2 \quad (17)$$

$u_{zz} \in L_2(G)$ Умножая неравенство (17) на $(2\pi)^{-1/2} \cdot (1 + |\lambda|^2)^3$ и интегрируя по λ от $-\infty$ до $+\infty$ получим

$$\begin{aligned} \|u\|_{W_2^{2,3}(G)}^2 &= (2\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} (1 + |\lambda|^2)^3 \|\hat{u}\|_{W_2^2(Q)}^2 d\lambda \leq \\ &\leq (2\pi)^{-1/2} \cdot c_2 \cdot \int_{-\infty}^{+\infty} (1 + |\lambda|^2)^3 \cdot \|\hat{f}\|_{W_2^1(Q)}^2 d\lambda = c_2 \|f\|_{W_2^{1,3}(G)}^2 \end{aligned} \quad (18)$$

из которой следует справедливость второй оценки теоремы 1 и существование обобщенного решения задачи (1)-(3) из пространства $W_2^{2,3}(Q)$. Теорема 1 доказана.

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SUBKRITIK IMMIGRATSIYALI GALTON-VATSON TARMOQLANUVCHI TASODIFIY
JARAYONLARINING ERGODIKLIK XOSSALARI

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Annotatsiya. Maqolada tasodifiy jarayonlar nazariyasida muhim o‘rin tutadigan va keng amaliy tatbiqlarga ega bo‘lgan Galton-Vatson immigratsiyali tarmoqlanuvchi jarayoni ko‘rilgan. Jarayonning tarmoqlanish qonunining ikkinchi tartibli faktorial momenti $f''(1-)$ chekli bo‘lgan holda $\{u_j, j \in S\}$ sonlar $\{p_{ij}\}$ o‘tish ehtimolliklariga nisbatan invariant taqsimot tashkil etishi o‘rganilgan. Shuningdek, bu invariant taqsimotning matematik kutilmasining bahosi ko‘rsatilgan.

Kalit so‘zlar: immigratsiyali tarmoqlanuvchi jarayon, hosil qiluvchi funksiya, tasodifiy miqdor, o‘tish ehtimolligi, invariant taqsimot.

ЭРГОДИЧНЫЕ СВОЙСТВА И ИНВАРИАНТНЫЕ МЕРЫ ВЕТВЯЩИХСЯ
СЛУЧАЙНЫХ ПРОЦЕССОВ ГАЛЬТОНА-ВАТСОНА С НЕКРИТИЧНОЙ
ИММИГРАЦИЯМИ

Аннотация. В статье рассматривается ветвящийся процесс Галтона–Ватсона с иммиграцией, который занимает важное место в теории случайных процессов и имеет широкие практические применения. Получена, что при условии конечности факториального момента второго порядка закона ветвления $f''(1-)$ числа $\{u_j, j \in S\}$ образуют инвариантное распределение относительно переходных вероятностей $\{p_{ij}\}$. Также показано оценка математического ожидания данного инвариантного распределения.

Ключевые слова: ветвящийся процесс, производящая функция, случайная величина, вероятность перехода, инвариантное распределение.

ERGODIC PROPERTIES AND INVARIANT MEASURES OF GALTON-WATSON
BRANCHING RANDOM PROCESSES WITH NON-CRITICAL IMMIGRATION

Abstract. The paper considers the Galton–Watson branching process with immigration, which plays an important role in the theory of stochastic processes and has wide practical applications. It is shown that, under the condition that the second-order factorial moment of the branching law is finite $f''(1-)$, the numbers $\{u_j, j \in S\}$ form an invariant distribution with respect to the transition probabilities $\{p_{ij}\}$. In addition, an estimate of the mathematical expectation of this invariant distribution is obtained.

Key words: branching process, generating function, random variable, transition probability, invariant distribution.

Kirish. Ushbu maqolada Galton-Vatson immigratsiyali (GVI) tarmoqlanuvchi jarayonlari deb ataladigan tasodifiy jarayonlari qaralgan. Zarrachalar populyatsiyasining evolyutsiyasi nuqtai nazaridan GVI jarayonining formal ta‘rifini quyidagicha keltirish qulay. Boshlang‘ich vaqtda sistemada $i \in \mathbb{N}$ ta zarracha mavjud bo‘lsin. Har bir zarracha birlik vaqt davomiyligida umr ko‘rsin. Hayotining so‘nggida bu zarracha ξ tasodifiy sondagi avlodni mos ravishda $P(\xi = k) =: p_k$ taqsimot qonuni bilan hosil qiladi, bu yerda $k \in \mathbb{N}_0$. Shuningdek ayni vaqtda, tizimga har birlik vaqt davomida biror $P(\eta = j) =: h_j$ qonunga ega bo‘lgan tasodifiy sondagi immigrant zarrachalar kirib kelsin. Kelgusida har bir zarracha boshqa zarrachalar kabi bir birlik umr ko‘radi va umrining oxirida boshqa zarrachalarga bog‘liq bo‘lmagan holda $\{p_k\}$ reproduktiv qonunga ko‘ra tasodifiy sondagi avlod qoldiradi.

Shunday qilib, vaqtning ixtiyoriy $n \in \mathbb{N}_0$ momentida tizimdagi zarrachalar soni X_n quyidagi rekurrent munosabat orqali aniqlanishi mumkin:

$$X_0 = i, \quad X_{n+1} = \xi_{n1} + \xi_{n2} + \dots + \xi_{nX_n} + \eta_{n+1}, \quad n \in \mathbf{N}_0, \quad (1)$$

bu yerda $\{\xi_{nk}\}$ – bir xil taqsimlangan o‘zaro bog‘liqsiz tasodifiy miqdorlar ketma-ketligi bo‘lib, u bevosita n -avlod k -zarrachasining avlodlari sifatida talqin qilinadi, η_n esa n -vaqtda kirib keluvchi ξ_{nk} dan bog‘liqsiz bo‘lgan bir xil taqsimlangan o‘zaro bog‘liqsiz tasodifiy miqdorlar. Bunda $\xi_{nk} \stackrel{d}{=} \xi$ miqdorlar barcha n va k lar uchun umumiy $p_k = P(\xi = k)$ taqsimotga, $\eta_m \stackrel{d}{=} \eta$ miqdorlar esa barcha m lar uchun $P(\eta = j) =: h_j$ taqsimotga ega.

GVI jarayonining yuqoridagi ta’rifi birinchi marta 1965 yilda S.Xitkot [1] tomonidan berilgan (shuningdek [2] ga qarang). Keyinchalik, hozirgi kunga qadar ma’lum bo‘lgan bir qator ishlar jarayonning strukturaviy va asimptotik xususiyatlariga bag‘ishlangan. Shu munosabat bilan, xususan, A.Barbour va A.Pakes [3], E.Seneta [4]–[7], A.Pakes [8], [9]–[12], S.Pommerenke [13] va boshqa mualliflarning ishlarini sanab o‘tish mumkin.

O‘tish ehtimolligining aniqroq shakldagi asimptotik ko‘rinishlari, shuningdek, zanjirning ergodiklik xossalari A.Imomovning [15], [16], [17] ishlarida topildi.

Tadqiqot obyekti va qo‘llanilgan metodlar. Tarmoqlanuvchi tasodifiy jarayonlarni o‘rganishda hosil qiluvchi funksiyalar muhim o‘rin tutadi. Quyidagi hosil qiluvchi funksiyalarni kiritaylik:

$$h(s) := \sum_{j \in \mathbf{N}_0} h_j s^j \quad \text{va} \quad f(s) := \sum_{j \in \mathbf{N}_0} p_j s^j$$

bunda $s \in [0,1)$. Tarmoqlanuvchi jarayon $\{X_n\}$ mumkin bo‘lgan holatlari fazosi $S \subset \mathbf{N}_0$ bilan berilgan bir jinsli Markov zanjirini aniqlaydi. Bu zanjirning i holatdan j holatga n qadamda o‘tish ehtimolliklarini har qanday k uchun quyidagicha belgilaymiz:

$$p_{ij}^{(n)} := P(X_{k+n} = j | X_k = i) = P(X_n = j | X_0 = i).$$

Kelgusida biz agar $\{X_n\}$ zanjirning n qadam oldingi holati $i \in S$ bo‘lsa, zanjir holatining shartli ehtimolligini ifodalashda $P_i(*)$ belgilashdan foydalanamiz. Masalan, $p_{ij}^{(n)}$ o‘tish ehtimolligi $P_i(X_n = j) = p_{ij}^{(n)}$ bilan belgilanadi. GVI jarayonining hosil qiluvchi funksiyasini foydalanish uchun qulay bo‘lgan quyidagi

$$P_n^{(i)}(s) = [f_n(s)]^i \prod_{k=0}^{n-1} h(f_k(s)), \quad (2)$$

ko‘rinishda aniqlab olamiz.

Yuqoridagi (2) tenglikni $s=1$ nuqtada to‘g‘ridan-to‘g‘ri differensiallasak, n -momentdagi zarrachalar sonining matematik kutilmasi uchun quyidagi ifodani topamiz:

$$E_i X_n = \sum_{k \in S} k p_{ik}^{(n)} = \begin{cases} \left(\frac{a}{m-1} + i \right) m^n - \frac{a}{m-1}, & m \neq 1, \\ an + i, & m = 1, \end{cases}$$

bu yerda $m = f'(1-)$ musbat parametr bo‘lib, u bitta aborigen avlodining o‘rtacha soni, $a = h'(1-)$ son esa birlik vaqt davomida tizimga kirib kelgan immigrant-zarralarning o‘rtacha soni. E_i belgi P_i ehtimollikka nisbatan matematik kutilmani ifodalaydi.

$\{X_n\}$ jarayon m parametrning qiymatiga bog‘liq ravishda tasniflanadi. Agar $m < 1$, $m = 1$ va $m > 1$ bo‘lsa, jarayon mos ravishda subkritik, kritik va superkritik deb ataladi.

Ushbu ishda analitik usullardan hosil qiluvchi funksiyalar usuli qo‘llanilgan.

Natijalar va muhokamalar. [21] maqolada $p_{ij}^{(n)}$ o‘tish ehtimolliklarining umumiy asimptotik xususiyatlari tekshirilgan. Ushbu maqolada $\{p_{0j}^{(n)} / p_{00}^{(n)}\}$ ketma-ketlikning monoton o‘svuvchi ekanligi va yuqoridan chegaralanganligi ko‘rsatilgan. Shuningdek, ketma-ketlik monoton o‘sgan holda chekli nomanfiy u_j limit qiymatga yaqinlashishi aniqlashtirilgan va

$$\frac{p_{0j}^{(n)}}{p_{00}^{(n)}} \uparrow u_j < \infty, \quad n \rightarrow \infty. \quad (3)$$

munosabat keltirilgan.

Nokritik ($m \neq 1$) holda, $R_n(s) = 1 - f_n(s)$ funksiya uchun [19] maqolada ushbu

$$R_n(s) = A(s)\beta^n(1 + o(1)), \quad n \rightarrow \infty, \quad (4)$$

bu yerda

$$\frac{1}{A(s)} = \frac{1}{q-s} + \gamma$$

va γ miqdor β va $f''(q)$ sonlarga bog'liq bo'lgan son, munosabat olingan. Bu miqdorning aniq qiymati ilk bor [20] ishda quyidagi ko'rinishda topilgan:

$$\gamma = \frac{f''(q)}{2\beta(1-\beta)}.$$

Yuqoridagi (4) formuladan $s=0$ bo'lgan holda $n \rightarrow \infty$ uchun ushbu $P(n < H < \infty) \sim K_q \beta^n$ asimptotik formulani olamiz, bunda

$$K_q \equiv A(0) = \frac{q}{1-q\gamma}$$

umumlashgan Kolmogorov doimiysi deb ataladi; qarang [20].

Kritik jarayonlar uchun ham yuqoridagi kabi asimptotik munosabatni hosil qilish mumkin. Bu holda $q=1$ va agar zarrachalar ko'payishi qonunining dispersiyasi chekli bo'lsa, u holda barcha $s \in [0,1)$ qiymatlar uchun $n \rightarrow \infty$ da ushbu $R_n(s) = O(1/n)$ asimptotik baho o'rinli bo'ladi; qarang [10].

[21] da $\{u_j, j \in S\}$ sonlar subkritik GVI jarayonlari uchun invariant taqsimot tashkil etishi ko'rsatilgan. Endi bu invariant taqsimotning matematik kutilmasini baholaymiz.

Teorema 1. Nokritik GVI tarmoqlanuvchi jarayonini qaraymiz. Agar subkritik hol uchun $f''(1-) < \infty$ bo'lsa, u holda $\{u_j, j \in S\}$ sonlar $\{p_{ij}\}$ o'tish ehtimolliklariga nisbatan invariant taqsimot tashkil etadi. Bu taqsimotning matematik kutilmasi uchun quyidagi baho o'rinli:

$$U'(1) = \frac{a}{1-\beta} U(1), \quad (5)$$

bunda

$$\frac{\sigma}{h_0} < U(1) = \exp \left\{ C \frac{K_q - A(1)}{1-\beta} \right\} < \infty,$$

bu yerda C o'zgarmas son.

Isbot. Bizni qiziqtirayotgan $U(s)$ funksiya (3) tasdiqda berilgan $\{u_j\}$ sonlar to'plami bilan bir qiymatli aniqlanishi tufayli quyidagi hosil qiluvchi funksiyani qaraymiz:

$$U_n(s) = \sum_{j \in N_0} \frac{p_{0j}^{(n)}}{p_{00}^{(n)}} s^j = \frac{1}{P_n(0)} P_n(s) = \prod_{k=0}^{n-1} \frac{h(f_k(s))}{h(f_k(0))}, \quad (6)$$

Dastlab, yuqorida keltirilgan (6) tenglikdan quyidagi munosabatlarni yozib olamiz:

$$\ln U_N(s) = \sum_{n=0}^{N-1} \ln \frac{h(f_n(s))}{h(f_n(0))} = - \sum_{n=0}^{N-1} \ln \frac{h(f_n(0))}{h(f_n(s))}. \quad (7)$$

Oxirgi tenglikning o'ng tomonidagi yig'indi belgisi ostida $\beta < \alpha$ munosabat bilan bog'langan barcha musbat miqdorlar uchun o'rinli bo'lgan ushbu

$$\frac{\beta - \alpha}{\beta} < \ln \frac{\beta}{\alpha} < \frac{\beta - \alpha}{\alpha} \quad (8)$$

tengsizlikni qo'llaymiz. Ushbu

$$D_n^{(y)}(s) := \frac{h(f_n(s)) - h(f_n(0))}{h(f_n(y))}, \quad y \in \{0, s\}$$

belgilashni kiritamiz va (8) tengsizlikka ko'ra

$$\sum_{n=0}^{N-1} D_n^{(s)}(s) < \ln U_N(s) < \sum_{n=0}^{N-1} D_n^{(0)}(s) \quad (9)$$

tengsizliklarga ega bo'lamiz, bunda $0 < h(f_n(0)) < h(f_n(s))$ ekanligidan foydalandik. O'rta qiymat haqidagi teorema ko'ra

$$h(f_n(s)) - h(f_n(0)) = h'(\theta_n(s))(f_n(s) - f_n(0)),$$

bu yerda $f_n(0) < \theta_n(s) < f_n(s)$ va $n \rightarrow \infty$ bo'lganda $\theta_n(s) \rightarrow q$. U holda (9) tengsizliklarning o'ng tomonidagi yig'indi belgisi ostidagi ifodani quyidagi ko'rinishda yozib olishimiz mumkin:

$$D_n^{(0)}(s) = d_n(s)(f_n(s) - f_n(0)), \quad (10)$$

bunda

$$d_n(s) = \frac{h'(\theta_n(s))}{h(f_n(0))} \rightarrow \frac{h'(q)}{h(q)}, \quad n \rightarrow \infty.$$

Dastlab, barcha $n \in \mathbb{N}$ nomerlar uchun ushbu

$$d_n(s) < \frac{h'(\theta_n(s))}{h(f_n(0))} < \frac{h'(f_n(s))}{h(0)} < \frac{a}{h_0} \quad (11)$$

tekis bahoga ega bo'lamiz. Ikkinchi tomondan $R_n(s) = 1 - f_n(s)$ belgilashga ko'ra, (4) formuladan

$$f_n(s) - f_n(0) = [K_q - A(s)]\beta^n(1 + o(1)), \quad n \rightarrow \infty \quad (12)$$

munosabatni hosil qilamiz, bu yerda $A(s)$ funksiya $\frac{1}{A(s)} = \frac{1}{q-s} + \gamma$ tenglik bilan aniqlangan va

$K_q = A(0)$ – Kolmogorov doimiysi. Endi (10) – (12) munosabatlarning kombinatsiyasidan

$$D_n^{(0)}(s) < \frac{a}{h_0} [K_q - A(s)]\beta^n(1 + o(1)), \quad n \rightarrow \infty$$

asimptotik munosabat kelib chiqadi. U holda $N \rightarrow \infty$ bo'lganda

$$\sum_{n=0}^{N-1} D_n^{(0)}(s) < \frac{a}{h_0} [K_q - A(s)] \sum_{n=0}^{N-1} \beta^n(1 + o(1)). \quad (13)$$

Ravshanki,

$$\sum_{n=0}^{N-1} \beta^n(1 + o(1)) = \frac{1 - \beta^N}{1 - \beta}(1 + r(N)), \quad (14)$$

bunda $N \rightarrow \infty$ bo'lganda $r(N) \rightarrow \text{const}$.

O'z navbatida yuqoridagi mulohazalar yordamida (9) tengsizliklarning chap tomonidagi yig'indini $N \rightarrow \infty$ da quyidan baholaymiz:

$$\sum_{n=0}^{N-1} D_n^{(s)}(s) > h_0 [K_q - A(s)] \sum_{n=0}^{N-1} \beta^n(1 + o(1)). \quad (15)$$

Topilgan (13) – (15) munosabatlardan $N \rightarrow \infty$ bo'lganda quyidagi asimptotik bahoni topamiz:

$$h_0 C_1 \leq \frac{(1 - \beta) \ln U_N(s)}{K_q - A(s)} \leq \frac{a}{h_0} C_2,$$

bu yerda C_1 va C_2 – o'zgarmas sonlar. Bu bahodan, quyidagi natijani olamiz:

$$\ln U(s) = C \frac{K_q - A(s)}{1 - \beta},$$

bu yerda C o'zgarmas son. Shunday qilib, $A(1) < \infty$ ekanligidan,

$$U(1) = \sum_{j \in S} u_j = \exp \left\{ C \frac{K_q - A(1)}{1 - \beta} \right\} < \infty. \quad (16)$$

Hosil qiluvchi funksiya $U(s) = \sum_{j \in S} u_j s^j$ ushbu

$$\sigma \cdot U(s) = h(s)U(f(s)) \quad (17)$$

funksional tenglamani qanoatlantiradi, bu yerda esa $\sigma = h(q)$, qarang [18]. Yuqoridagi (17) tenglamadan $\sigma u_j = \sum_{i \in S} u_i p_{ij}$ tenglik kelib chiqadi. Ta'rifga ko'ra $u_0 = U(0) = 1$. U holda quyidagi munosabatlarni yozib olamiz:

$$\begin{aligned}\sigma &= \sigma u_0 = \sum_{i \in S} u_i p_{i0} \\ &= \sum_{i \in S} u_i h_0 p_0^i = h_0 U(p_0) < h_0 U(1).\end{aligned}$$

Oxirgi tengsizlikdan va (16) bahodan, ushbu

$$\frac{\sigma}{h_0} < \sum_{j \in S} u_j = \exp \left\{ C \frac{K_q - A(1)}{1 - \beta} \right\} < \infty \quad (18)$$

baho kelib chiqadi. Olingan (18) baho $\{u_j, j \in S\}$ sonlar $\{p_{ij}\}$ o'tish ehtimolliklariga nisbatan σ -invariant taqsimot tashkil etishini bildiradi. Ikkinchi tomondan, (17) tenglamani differensiallab, quyidagini topamiz:

$$U'(s) = h'(s)U(f(s)) + h(s)U'(f(s))f'(s).$$

Topilgan tenglik, $s = 1$ qiymatda bizga ushbu

$$U'(1) = \frac{a}{1 - \beta} U(1)$$

tenglikni beradi. Bu tenglik bilan (18) bahoni birgalikda qarab, (5) tasdiqni hosil qilamiz.

Teorema isbot bo'ldi.

Xulosa. Ushbu ishda supkritik Galton-Vatson immigratsiyali tarmoqlanuvchi jarayoni qaraldi. Subkritik hol uchun ikkinchi tartibli faktorial momenti $f''(1-) < \infty$ bo'lganda $\{u_j, j \in S\}$ sonlar $\{p_{ij}\}$ o'tish ehtimolliklariga nisbatan invariant taqsimot tashkil etishi aniqlandi. Shuningdek, bu taqsimotning matematik kutilmasi uchun baho topildi.

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AN INVERSE PROBLEM OF DETERMINING A COEFFICIENT FOR A TIME-FRACTIONAL DIFFUSION-WAVE EQUATION INVOLVING A CAPUTO DERIVATIVE

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Abstract. This work investigates an inverse problem for determining a time-dependent coefficient in a fractional diffusion-wave equation with a Caputo derivative. In this case, an initial boundary value problem was set for the fractional diffusion-wave equation, and an additional condition was given for the inverse problem of determining the coefficient from this equation. First of all, it was considered the initial boundary value problem. By the Fourier method, this problem is reduced to equivalent integral equations. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved.

Keywords: Time-fractional diffusion equation, Caputo fractional derivative, inverse problem, Fourier series, Banach fixed point theorem.

KAPUTO HOSILASINI O'Z ICHIGA OLGAN VAQT BO'YICHA KASRLI DIFFUZIYA-TO'LIQIN TENGLAMASI UCHUN KOEFFITSIYENTNI ANIQLASHNING TESKARI MASALASI

Annotatsiya. Ushbu ishda Kaputo hosilali kasr tartibli diffuzion to'liqin tenglamasida vaqtga bog'liq ko'effitsiyentni aniqlash uchun teskari masala o'rganilgan. Bunda kasr diffuzion to'liqin tenglamasi uchun boshlang'ich chegaraviy masala qo'yildi va bu tenglamadan ko'effitsiyentni aniqlash uchun teskari masalaga qo'shimcha shart berildi. Avvalo, boshlang'ich chegaraviy masala deb qaraldi. Furiye usuli bilan bu masala ekvivalent integral tenglamalarga keltiriladi. Teskari masala Volterra tipidagi tenglamaning ekvivalent integraliga keltiriladi. Bu tenglamani yechish uchun qisqartirilgan akslantirish prinsipidan foydalaniladi. Lokal mavjudlik va global yagonalik natijalari isbotlangan.

Kalit so'zlar: Vaqt bo'yicha kasrli diffuziya tenglamasi, Kaputo kasrli hosilasi, teskari masala, Furiye qatori, Banaxning qo'zg'almas nuqtalar teoremasi.

ОБРАТНАЯ ЗАДАЧА ОПРЕДЕЛЕНИЯ КОЭФФИЦИЕНТА ДЛЯ ДИФФУЗИОННО-ВОЛНОВОГО УРАВНЕНИЯ С ДРОБНОЙ ПРОИЗВОДНОЙ КАПУТО

Аннотация. В данной работе исследуется обратная задача для определения коэффициента, зависящего от времени, в дробном диффузионно-волновом уравнении с производной Капуто. В этом случае для дробного диффузионно-волнового уравнения была поставлена начальная краевая задача, а для обратной задачи определения коэффициента из этого уравнения было дано дополнительное условие. Сначала рассматривалась начальная краевая задача. Методом Фурье эта задача сводится к эквивалентным интегральным уравнениям. Обратная задача сводится к эквивалентному интегралу уравнения типа Вольтерра. Для решения этого уравнения используется принцип сжатого отображения. Доказаны результаты локального существования и глобальной единственности.

Ключевые слова: Дробное уравнение диффузии во времени, дробная производная Капуто, обратная задача, ряд Фурье, теорема Банаха относительно фиксированной точки.

Introduction. Recently, a large number of applied problems have been formulated on fractional differential equations and consequently considerable attention has been given to the solutions of those equations. They allow for a more accurate representation of the time-dependent behavior of complex materials and processes, such as viscoelasticity, dynamic processes in fractal structures, control system theory, electrochemistry, and diffusion phenomena [1,2,3]. Due to the fractional differentiation operation, these equations account for crucial properties like memory effects (dependence on past states) and non-locality. This research is devoted to studying solutions to initial-boundary value problems for the fractional diffusion-wave equation, which generalizes the classical first-order diffusion and wave equations and is used to model subdiffusive, superdiffusive, and dispersive processes.

Inverse problems for fractional differential equations, including both wave and diffusion types, remain relatively underexplored. In the existing literature, the most commonly studied topics include the determination of the time-fractional derivative order, linear inverse problems for recovering source terms [4,5,6], and certain nonlinear inverse coefficient problems within initial-boundary value problems for fractional diffusion-wave equations under various overdetermination conditions (see also references cited therein).

In the work by Turdiev [7], the author investigates the solvability of the Cauchy problem for a time-space fractional diffusion-wave equation with a variable coefficient. The study employs spectral methods and fractional calculus techniques to analyze the direct problem and establishes rigorous results on existence, uniqueness, and stability of solutions under appropriate initial and boundary conditions. In the work of Durdiev and Turdiev [8], the authors investigate an inverse coefficient problem for a wave equation involving the generalized Riemann–Liouville time derivative. The study focuses on determining an unknown time-dependent coefficient from initial-boundary and additional overdetermination conditions.

Wave processes taking into account the history of the medium are described by hyperbolic equations with integral terms of convolution type. The investigation of inverse problems of determining the kernel of integral operators in hyperbolic integro-differential equations based on some information about the wave field plays an important role in studying the structure and properties of the medium. Inverse problems for integro-differential equations are a relatively new and rapidly developing direction of the theory of inverse problems. Coefficient inverse problems are devoted to works [9], [10]. Inverse problems for integro-differential heat and wave equation were investigated in [11] and there shown the well-posedness of problem in suitable functional spaces. In work [12], inverse problem of determining kernel of the equation of viscoelasticity are investigated. In work [13] are devoted to the investigation of inverse problem for the integro-differential equation of acoustics.

In this paper, we investigate the local existence and global uniqueness of an inverse problem of determining time-dependent coefficient in the time-fractional diffusion wave equation with initial-boundary and overdetermination conditions.

In the domain $\Omega_T := \{(x, t) : 0 \leq x \leq l, 0 \leq t \leq T\}$ consider the time-fractional diffusion wave equation

$$\partial_t^\alpha u(x, t) - u_{xx}(x, t) + q(t)u(x, t) = f(x, t), \quad (x, t) \in \Omega_T, \quad (1)$$

with initial and boundary conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in [0, l], \quad (2)$$

$$u(0, t) = u(l, t) = 0, \quad 0 \leq t \leq T, \quad (3)$$

where ∂_t^α is the Caputo fractional derivative of order $1 < \alpha < 2$ in the time variable (see definition 1, 2 in preliminaries) and $\varphi(x)$, $\psi(x)$, $f(x, t)$ are given smooth functions.

We pose the inverse problem as follows: find the function $q(t)$, $q \geq 0$ in (1), if the solution of the initial-boundary problem (1)-(3) satisfies condition:

$$\int_0^l w(x)u(x, t)dx = g(t), \quad 0 \leq t \leq T, \quad (4)$$

$g(t)$ is a given function.

The functions $\varphi(x)$, $\psi(x)$, $f(x, t)$ and $g(t)$ satisfy the following assumptions

A1) $\{\varphi, \psi\} \in C^3[0, l]$, $\{\varphi^{(4)}, \psi^{(4)}\} \in L_2(0, l)$, $\varphi(0) = \varphi(l) = 0$, $\psi(0) = \psi(l) = 0$, $\varphi'(0) = \varphi'(l) = 0$, $\psi'(0) = \psi'(l) = 0$;

A2) $f(x, \cdot) \in C[0, T]$ and for $t \in [0, T]$, $f(\cdot, t) \in C^3[0, l]$, $f^{(4)}(\cdot, t) \in L_2(0, l)$, $f(0, t) = f(l, t) = 0$, $f_{xx}(0, t) = f_{xx}(l, t) = 0$;

A3) $w(x) \in C^2[0, l]$ and $w(0) = w(l) = 0$ and $w'(0) = w'(l) = 0$;

A4) $\partial_t^\alpha g(t) \in C[0, T]$ and $|g(t)| \geq g_0 > 0$, g_0 is a given number, $\int_0^l w(x)\varphi(x)dx = g(0)$, $\int_0^l w(x)\psi(x)dx = g'(0)$.

The article is organized as follows: In Section 2, we give some basic definitions and results needed in the sequel. In Section 3, the existence and uniqueness of the solution to direct problem (1)-(3) are obtained. Here also the stability estimate for this solution is given. Section 4 is devoted to the solving of inverse problem (1)-(4).

Preliminaries. In this section, we present some useful definitions and results, which will be used in the future.

Definition 1. The Riemann-Liouville fractional integral of order $n-1 < \alpha < n$ for an integrable function $h(t)$ is defined by

$$I_t^{n-\alpha} h(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > 0.$$

Definition 2. The Riemann-Liouville fractional derivative of order $n-1 < \alpha < n$ of the functions $h(t) \in AC^n[0, T]$ [see [1] pp. 69-76] is defined by

$$\partial_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > 0.$$

Two parameter Mittag-Leffler function. The two parameter Mittag-Leffler function $E_{\alpha, \beta}(z)$ is defined by the following series:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$ denote the real part of the complex number α , $\Gamma(\cdot)$ is Euler's Gamma function. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications. A very interesting work that meets many results about this function is due to Kilbas et al. (see [[1], pp. 42-44]).

Proposition 1. Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1+|z|}, \quad \kappa \leq \arg(z) \leq \pi.$$

For the proof, we refer to [[1], pp. 40-45] for example.

Definition 3. We consider the weighted spaces of continuous functions [[1], pp.4-5, 162-163].

$$\begin{aligned} C^\alpha[0, T] &= \{h(t) : \partial_t^\alpha h(t) \in C_\gamma[0, T], \quad 1 < \alpha < 2\}, \\ C^{2, \alpha}(\Omega_T) &= \{u(x, t) : u(\cdot, t) \in C^2(0, 1); \quad t \in [0, T] \text{ and} \\ &\quad \partial_t^\alpha u(x, \cdot) \in C_\gamma(0, T]; \quad x \in [0, 1], \quad 1 < \alpha < 2\}, \\ C^0[a, b] &= C[a, b], \end{aligned}$$

with the norms

$$\|f\|_{C^n} = \sum_{k=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_C.$$

Lemma 1. [[14], p.188] Suppose $b \geq 0$, $\alpha > 0$ and $a(t)$ nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds$$

on this interval; then

$$u(t) \leq a(t) + b \Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(b \Gamma(\alpha)(t-s)^\alpha) a(s) ds.$$

Lemma 2. [[14], p.189] Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ nonnegative function locally integrable on $0 \leq t < T$ and suppose $t^{\gamma-1} u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s) ds$$

a.e. in $(0, T)$; then

$$u(t) \leq a(t) E_{\alpha, \gamma} \left((b \Gamma(\alpha))^{\frac{1}{\alpha+1}} t \right),$$

where

$$E_{\alpha, \gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}, \quad c_0 = 1, \quad \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$$

for $m \geq 0$. As $t \rightarrow +\infty$ $E_{\alpha, \gamma}(t) = O\left(t^{\frac{1}{\alpha+\gamma-1}} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}}\right)\right)$.

In the next section we will deal with finding the solution of the initial-boundary problem (1)-(3).

Existence and uniqueness result. By applying the Fourier method, the solution $u(x, t)$ of the problem (1)-(3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(t) X_n(x), \quad (5)$$

where

$$X_n(x) = \sqrt{\frac{2}{l}} \sin(\lambda_n x), \quad \lambda_n = \frac{\pi n}{l}, \quad n = 1, 2, 3, \dots \quad (6)$$

The coefficients $u_n(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions $X_n(x)$. The scalar product in $L_2[0, 1]$ is defined by $(f, g) = \int_0^1 f(x)g(x)dx$. Let us note the expansion coefficients of $\varphi(x)$, $\psi(x)$ and $f(x, t)$ in the eigenfunctions of (6) for $n \geq 1$ are defined respectively by

$$\begin{aligned} (f(x, t), X_n(x)) &= f_n(t), \\ (\varphi(x), X_n(x)) &= \varphi_n, \quad (\psi(x), X_n(x)) = \psi_n, \quad n = 1, 2, \dots \end{aligned} \quad (7)$$

We obtain in view of (1) and with $(u(x, t), X_n(x)) = \int_0^l u(x, t) X_n(x) dx = u_n(t)$, and we may write

$$\partial_t^\alpha u_n(t) + \lambda_n^2 u_n(t) + q(t) u_n(t) = f_n(t), \quad (8)$$

$$u_n(0) = \varphi_n, \quad u_n'(0) = \psi_n. \quad (9)$$

We suppose that $f_n(t) \in C[0, T]$. Then, by property 3.1(a) (see [[1], p.p. 230-234]), (8), (9) is equivalent in the space $C^\alpha[0, T]$ to the following Volterra integral equation:

$$\begin{aligned} u_n(t) &= \varphi_n E_{\alpha, 1}(-\lambda_n^2 t^\alpha) + \\ &+ \psi_n t E_{\alpha, 2}(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n^2 (t-\tau)^\alpha) f_n(\tau) d\tau - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n^2 (t-\tau)^\alpha) q(\tau) u_n(\tau) d\tau. \end{aligned} \quad (10)$$

First we prove the following assertions:

Lemma 3. For any $n \in \mathbb{N}$, $q \in C[0, T]$ we have the estimates

$$\begin{aligned} |u_n(t)| &\leq C(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\|) E_{\alpha, 1} \left((C \|q\|_{C[0, T]} \Gamma(\alpha))^{\frac{1}{\alpha}} t \right), \quad t \in [0, T], \\ |\partial_t^\alpha u_n(t)| &\leq \|f_n\| + C(\lambda_n^2 + \|q\|_{C[0, T]}) \times \left(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\| \right) E_{\alpha, 1} \left((C \|q\|_{C[0, T]} \Gamma(\alpha))^{\frac{1}{\alpha}} t \right), \quad t \in [0, T]. \end{aligned}$$

Proof. In the integral equation (10), taking into account conditions A1), A2), we estimate

$$\begin{aligned} |u_n(t)| &\leq |\varphi_n| E_{\alpha, \alpha}(-\lambda_n^2 t^\alpha) + |\psi_n| t E_{\alpha, \alpha-1}(-\lambda_n^2 t^\alpha) + \left| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n^2 (t-\tau)^\alpha) f_n(\tau) d\tau \right| + \\ &+ \left| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n^2 (t-\tau)^\alpha) q(\tau) u_n(\tau) d\tau \right| \leq C(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\|) + C \|q\|_{C[0, T]} \int_0^t (t-\tau)^{\alpha-1} |u_n(\tau)| d\tau. \end{aligned}$$

Next, according to Lemma 2, we have

$$|u_n(t)| \leq C \left(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\| \right) E_{\alpha, 1} \left((C \|q\|_{C[0, T]} \Gamma(\alpha))^{\frac{1}{\alpha}} t \right). \quad (11)$$

We get the second part of the lemma 3, from equation (8) and the first estimate of Lemma 3

$$|\partial_t^\alpha u_n(t)| \leq \|f_n\| + C(\lambda_n^2 + \|q\|_{C[0,T]}) \times \left(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\| \right) E_{\alpha,1} \left(\left(C\|q\|_{C[0,T]} \Gamma(\alpha) \right)^{\frac{1}{\alpha}} t \right). \quad (12)$$

From the last two inequalities we immediately obtain the estimates of lemma 3 for any $t \in [0, T]$.
Lemma proven.

Formally, from (5) by term-by-term differentiation we compose the series

$$\partial_t^\alpha u(x, t) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \partial_t^\alpha u_n(t) \sin(\lambda_n x), \quad (13)$$

$$u_{xx}(x, t) = \sqrt{\frac{2}{l}} \sum_{n=0}^{\infty} \lambda_n^2 u_n(t) \sin(\lambda_n x). \quad (14)$$

Let us prove the uniform convergence of series (5), (13) and (14) in the domain $\overline{\Omega}_T$. This series for any $(x, t) \in \overline{\Omega}_T$ is majorized by

$$C \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left(|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\| \right) E_{\alpha,1} \left(\left(C\|q\|_{C[0,T]} \Gamma(\alpha) \right)^{\frac{1}{\alpha}} T \right), \quad (15)$$

$$\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[\|f_n\| + C(\lambda_n^2 + \|q\|_{C[0,T]}) \left(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\| \right) \times E_{\alpha,1} \left(\left(C\|q\|_{C[0,T]} \Gamma(\alpha) \right)^{\frac{1}{\alpha}} t \right) \right], \quad (16)$$

$$C \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \lambda_n^2 \left(|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\| \right) E_{\alpha,1} \left(\left(C\|q\|_{C[0,T]} \Gamma(\alpha) \right)^{\frac{1}{\alpha}} T \right), \quad (17)$$

where $\overline{\Omega}_T := \{(x, t) : 0 \leq x \leq l, 0 \leq t \leq T\}$.

We hold the following auxiliary lemma.

Lemma 4. If the conditions A1), A2) are fulfilled then there are equalities

$$\varphi_n = \frac{1}{\lambda_n^3} \varphi_n^{(3)}, \quad \psi_n = \frac{1}{\lambda_n^3} \psi_n^{(3)}, \quad f_n = \frac{1}{\lambda_n^3} f_n^{(3)}, \quad (18)$$

where

$$\begin{aligned} \varphi_n^{(3)} &= \sqrt{\frac{2}{l}} \int_0^l \varphi^{(3)}(x) \cos(\lambda_n x) dx, & \psi_n^{(3)} &= \sqrt{\frac{2}{l}} \int_0^l \psi^{(3)}(x) \cos(\lambda_n x) dx, \\ f_n^{(3)} &= \sqrt{\frac{2}{l}} \int_0^l f^{(3)}(x) \cos(\lambda_n x) dx, \end{aligned}$$

with the following estimates:

$$\begin{aligned} \sum_{n=1}^{\infty} |\varphi_n^{(3)}|^2 &= \|\varphi^{(3)}\|_{L_2(0,l)}, & \sum_{n=1}^{\infty} |\psi_n^{(3)}|^2 &= \|\psi^{(3)}\|_{L_2(0,l)}, \\ \sum_{n=1}^{\infty} |f_n^{(3)}|^2 &= \|f^{(3)}\|_{L_2[0,T]}. \end{aligned} \quad (19)$$

If the functions $\varphi(x)$, $\psi(x)$ and $f(x, t)$ satisfy the conditions of Lemma 4, then due to representations (18) and (19) series (5), (13) and (14) converge uniformly in the rectangle $\overline{\Omega}_T$, therefore, function $u(x, t)$ satisfies relations (1)–(3).

Using the above results, we obtain the following assertion.

Theorem 1. Let $q(t) \in C[0, T]$, A1), A2) are satisfied, then there exists a unique solution of the direct problem (1)–(3) $u(x, t) \in C^{2,\alpha}(\overline{\Omega}_T)$.

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (10) and the solution of this equation with perturbed functions $\tilde{q}$, $\tilde{\varphi}_n$, $\tilde{\psi}_n$ and $\tilde{f}_n$. Let $\tilde{u}_n(t)$ be solution of the integral equation (10) corresponding to the functions $\tilde{q}$, $\tilde{\varphi}_n$, $\tilde{\psi}_n$, $\tilde{f}_n$; i.e.,

$$\begin{aligned} \tilde{u}_n(t) = & \tilde{\varphi}_n E_{\alpha,1}(-\lambda_n^2 t^\alpha) + \tilde{\psi}_n t E_{\alpha,2}(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \tilde{f}_n(\tau) d\tau \\ & - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \tilde{q}(\tau) \tilde{u}_n(\tau) d\tau. \end{aligned} \quad (20)$$

Composing the difference $u_n(t) - \tilde{u}_n(t)$ with the help of the equations (10), (20) and introducing the notations $\hat{u}_n(t) = u_n(t) - \tilde{u}_n(t)$, $\hat{q} = q(t) - \tilde{q}(t)$, $\hat{\varphi}_n(t) = \varphi_n(t) - \tilde{\varphi}_n(t)$, $\hat{\psi}_n(t) = \psi_n(t) - \tilde{\psi}_n(t)$, $\hat{f}_n(t) = f_n(t) - \tilde{f}_n(t)$ we obtain the integral equation

$$\begin{aligned} \hat{u}_n(t) = & \hat{\varphi}_n E_{\alpha,1}(-\lambda_n^2 t^\alpha) + \hat{\psi}_n t E_{\alpha,2}(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \hat{f}_n(\tau) d\tau \\ & - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \hat{q}(\tau) u_n(\tau) d\tau - \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \tilde{q}(\tau) \hat{u}_n(\tau) d\tau, \end{aligned} \quad (21)$$

from which, is derived the following linear integral inequality for $|\hat{u}_n(t)|$

$$\begin{aligned} |\hat{u}_n(t)| \leq & C(|\hat{\varphi}_n| + |\hat{\psi}_n| t + \frac{t^\alpha}{\alpha} \|\hat{f}_n\|) + C \frac{t^\alpha}{\alpha} \|\hat{q}\|_{C[0,T]} \left(|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\| \right) \\ & \times E_{\alpha,1} \left((C\|q\|_{C[0,T]}\Gamma(\alpha))^{\frac{1}{\alpha}} t \right) + C\|\tilde{q}\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha-1} |\hat{u}_n(\tau)| d\tau. \end{aligned}$$

Using the Lemma 2 from last inequality, we arrive at the estimate:

$$\begin{aligned} |\hat{u}_n(t)| \leq & C[|\hat{\varphi}_n| + |\hat{\psi}_n| t + \frac{t^\alpha}{\alpha} \|\hat{f}_n\| + \frac{t^\alpha}{\alpha} \|\hat{q}\|_{C[0,T]} (|\varphi_n| + |\psi_n| t + \frac{t^\alpha}{\alpha} \|f_n\|)] \\ & \times E_{\alpha,1} \left((C\|q\|_{C[0,T]}\Gamma(\alpha))^{\frac{1}{\alpha}} t \right) + E_{\alpha,1} \left((C\|\tilde{q}\|_{C[0,T]}\Gamma(\alpha))^{\frac{1}{\alpha}} t \right). \end{aligned} \quad (22)$$

Indeed, the expression (22) is stability estimate for the solution to the problem (1)-(3). The uniqueness of this solution follows from (22).

Investigation of the inverse problem (1)-(4).

Let us now we investigate to inverse problem (1)-(4)

We integrate both sides of equation (1) in the interval from 0 to l with respect to x , we obtain

$$\int_0^l w(x) \{ \partial_t^\alpha u(x,t) - u_{xx} + q(t)u(x,t) \} dx = \int_0^l w(x) f(x,t) dx.$$

Applying ∂_t^α to the over-determination condition (4), we obtain the following equation

$$\partial_t^\alpha g(t) + q(t)g(t) - \int_0^l w''(x)u(x,t) dx = \int_0^l w(x)f(x,t) dx.$$

Using the above equality and condition A3), we obtain the following equality with respect to $q(t)$:

$$q(t) = \frac{1}{g(t)} \left(\int_0^l w(x)f(x,t) dx - \partial_t^\alpha g(t) + \int_0^l w''(x)u(x,t) dx \right).$$

The functions $u_n(t)$ depend on $q(t)$, i.e. $u_n(t,q)$. After simple converting, we get the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) + \frac{1}{g(t)} \sum_{n=1}^{\infty} w_n u_n(t,q), \quad (23)$$

where

$$w_n = \sqrt{\frac{2}{l}} \int_0^l w''(x) \sin(\lambda_n x) dx, \quad q_0(t) = \frac{1}{g(t)} \left(\int_0^l w(x)f(x,t) dx - \partial_t^\alpha g(t) \right).$$

We introduce an operator Ψ defining it by the right hand side of (23):

$$\Psi[q](t) = q_0(t) + \frac{1}{g_0} \sum_{n=1}^{\infty} w_n u_n(t, q). \quad (24)$$

Then, the equation (24) is written in more convenient form as

$$\Psi[q](t) = q(t). \quad (25)$$

Let

$$\|q_0\|_{C[0,T]} := \max_{t \in [0,T]} |q_0(t)| \leq \frac{1}{g_0} \left(\|w\|_{L_2(0,T)} \|f\|_{C(\bar{\Omega})} + \|g\|_{C[0,T]} \right).$$

Fix a number $r > 0$ and consider the ball

$$B(q_0, r) := \{q(t) : q(t) \in C[0, T], \|q - q_0\| \leq r\}.$$

Theorem 2. Let A1)-A4) are satisfied. Then there exists a number $T^* \in (0, T)$, such that there exists a unique solution $q(t) \in C[0, T^*]$ of the inverse problem (1)-(4).

Proof. Let us first prove that for an enough small $T > 0$ the operator Ψ maps the ball $B(q_0, r)$ implies that $\Psi[q](t) \in B(q_0, r)$. Indeed, for any continuous function $q(t)$, the function $\Psi[q](t)$ calculated using formula (25) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\begin{aligned} \|\Psi[q](t) - q_0(t)\| &\leq \frac{\|w\|_{L_2(0,T)}}{g_0} \\ &\times \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\|) E_{\alpha,1} \left((C \|q\|_{C[0,T]} \Gamma(\alpha))^{\frac{1}{\alpha}} T \right). \end{aligned}$$

Here we have used the estimate (11). In view of Lemmas 3 and 4 last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T , and the fact that the function $q(t)$ belongs to the ball $B(q_0, r)$ implies the inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$. Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\|q_0\|_{C[0,T]} + r$. Performing these replacements, we obtain the estimate

$$\begin{aligned} \|\Psi[q](t) - q_0(t)\| &\leq \frac{\|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\|) \\ &E_{\alpha,1} \left((C (\|q_0\|_{C[0,T]} + r) \Gamma(\alpha))^{\frac{1}{\alpha}} T \right). \end{aligned}$$

Therefore if by T_1 we denote the root of the equation (for T)

$$\frac{\|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\|) E_{\alpha,1} \left((C (\|q_0\|_{C[0,T]} + r) \Gamma(\alpha))^{\frac{1}{\alpha}} T \right) = r,$$

then $\|\Psi[q](t) - q_0(t)\| \leq r$ for $T \leq T_1$; i.e. $\Psi[q](t) \in B(q_0, r)$.

Now let us take any functions $q(t), \tilde{q}(t) \in B(q_0, r)$ and estimate the distance between their images $\Psi[q](t)$ and $\Psi[\tilde{q}](t)$ in the space $C[0, T]$. The function $\tilde{u}_n(t)$ corresponding to $\tilde{q}(t)$ satisfies the integral equation (20) with $\varphi_n = \tilde{\varphi}_n$, $\psi = \tilde{\psi}_n$ and $f_n = \tilde{f}_n$. Composing the difference $\Psi[q](t) - \Psi[\tilde{q}](t)$ with the help of equations (10), (20) and then estimating its norm, we obtain

$$\begin{aligned} \|\Psi[q](t) - \Psi[\tilde{q}](t)\| &\leq \frac{\|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} \|u_n(t, q) - \tilde{u}_n(t, \tilde{q})\| \leq \\ &\leq \frac{C \|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n| T + \frac{T^\alpha}{\alpha} \|f_n\|) E_{\alpha,1} \left((C \|q\|_{C[0,T]} \Gamma(\alpha))^{\frac{1}{\alpha}} T \right) \\ &\times E_{\alpha,1} \left((C \|\tilde{q}\|_{C[0,T]} \Gamma(\alpha))^{\frac{1}{\alpha}} T \right) \|\tilde{q}\|_{C[0,T]}. \end{aligned} \quad (26)$$

The functions $q(t)$ and $\tilde{q}(t)$ belong to the ball $B(q_0, r)$, and hence for each of these functions one has inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$. Note that the function on the right-hand side in inequality (26) at the factor

$\|q - \tilde{q}\|_{C[0,T]}$ is monotone increasing with $\|q\|_{C[0,T]}$, $\|\tilde{q}\|_{C[0,T]}$ and T . Consequently, replacing $\|q\|_{C[0,T]}$ and $\|\tilde{q}\|_{C[0,T]}$ in inequality (26) with $\|q_0\|_{C[0,T]} + r$ will only strengthen the inequality. This, we have

$$\begin{aligned} \|\Psi[q](t) - \Psi[\tilde{q}](t)\| &\leq \frac{C\|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n|T + \frac{T^\alpha}{\alpha} \|f_n\|) \\ &\quad \times \left(E_{\alpha,1} \left((C(\|q_0\|_{C[0,T]} + r)\Gamma(\alpha))^{\frac{1}{\alpha}} T \right) \right)^2 \|\tilde{q}\|_{C[0,T]}. \end{aligned}$$

Therefore, if T_2 is the root of the equation (for T)

$$\frac{C\|w\|_{L_2(0,T)}}{g_0} \sum_{n=1}^{\infty} (|\varphi_n| + |\psi_n|T + \frac{T^\alpha}{\alpha} \|f_n\|) \left(E_{\alpha,1} \left((C(\|q_0\|_{C[0,T]} + r)\Gamma(\alpha))^{\frac{1}{\alpha}} T \right) \right)^2 = 1$$

then for $T \in [0, T_2]$ the operator Ψ contracts the distance between the elements $q(t), \tilde{q}(t) \in B(q_0, r)$. Consequently, if we choose $T^* = \min(T_1, T_2)$ then the operator Ψ is a contraction in the ball $B(q_0, r)$. However, in accordance with the Banach theorem (see [[15], p.p. 87–97]), the operator Ψ has unique fixed point in the ball $B(q_0, r)$ i.e., there exists a unique solution of equation (25). Theorem 2 is proven.

Let T, l be positive fixed numbers. Consider the set D_{v_0} of the given functions (φ, ψ, g, f) for which all conditions from A1)-A4) are fulfilled and

$$\max\{\|\varphi\|_{C^3[0,T]}, \|\psi\|_{C^3[0,T]}, \|g\|_{C^\alpha[0,T]}, \|f\|_{C(\bar{\Omega}_T)}\} \leq v_0.$$

We denote by F_{v_1} the set of function $q(t)$ that for some $T > 0, l > 0$ satisfy the following condition $\|F\|_{C[0,T]} \leq v_1, v_1 > 0$.

Theorem 3. Let $(\varphi, \psi, g, f) \in D_{v_0}, (\tilde{\varphi}, \tilde{\psi}, \tilde{g}, \tilde{f}) \in D_{v_0}$ and $q \in F_{v_1}, \tilde{q} \in F_{v_1}$. Then, for solution of the inverse problem (1)-(4) the following stability estimate holds:

$$\begin{aligned} \|q - \tilde{q}\|_{C[0,T]} &\leq \rho [\|\varphi - \tilde{\varphi}\|_{C[0,T]} \\ &\quad + \|\psi - \tilde{\psi}\|_{C[0,T]} + \|g - \tilde{g}\|_{C^\alpha[0,T]} + \|f - \tilde{f}\|_{C(\bar{\Omega}_T)}], \end{aligned} \quad (27)$$

where the constant ρ depends only on v_0, v_1, T, l, α , and $\Gamma(\alpha), B(\alpha, 1-\gamma)$.

Proof. To prove this theorem, using (23) we write down the equations for $\tilde{q}(t)$ and compose the difference $\hat{q} = q(t) - \tilde{q}(t)$. Then after evaluating this expression and using estimates $u_n(t), \tilde{u}_n(t)$, we obtain following estimates

$$\begin{aligned} \|q - \tilde{q}\|_{C[0,T]} &= \max_{0 \leq t \leq T} \left| \frac{1}{g(t)} \left(\int_0^l w(x) f(x, t) dx - \partial_t^\alpha g(t) \right) + \frac{1}{g(t)} \int_0^l w''(x) u(x, t) dx \right. \\ &\quad \left. - \frac{1}{\tilde{g}(t)} \left(\int_0^l w(x) \tilde{f}(x, t) dx - \partial_t^\alpha \tilde{g}(t) \right) + \frac{1}{\tilde{g}(t)} \int_0^l w''(x) \tilde{u}(x, t) dx \right| \leq \max_{0 \leq t \leq T} \left\{ \frac{w_0}{g_0^2} \left| \int_0^l [\tilde{g}(t)(f(x, t) - \tilde{f}(x, t)) + \tilde{f}(x, t)(g(t) - \tilde{g}(t))] dx \right| \right. \\ &\quad \left. + \int_0^l [\tilde{g}(t)(\partial_t^\alpha g(t) - \partial_t^\alpha \tilde{g}(t)) + \partial_t^\alpha \tilde{g}(t)(g(t) - \tilde{g}(t))] dx \right\} + \max_{0 \leq t \leq T} \left\{ \frac{w_0}{g_0^2} \left| \int_0^l [\tilde{g}(t)(u(x, t) - \tilde{u}(x, t)) + \tilde{u}(x, t)(g(t) - \tilde{g}(t))] dx \right| \right\} \leq \\ &\leq \rho_0 (\|\varphi - \tilde{\varphi}\| + \|\psi - \tilde{\psi}\| + \|f - \tilde{f}\| + \|g - \tilde{g}\| + \|\partial_t^\alpha g - \partial_t^\alpha \tilde{g}\|) + \rho_1 \int_0^t (t-\tau)^{\alpha-1} \|q(\tau) - \tilde{q}(\tau)\|_{C[0,T]} d\tau, \quad t \in [0, T], \end{aligned} \quad (28)$$

where ρ_0, ρ_1 depends only on v_0, v_1, T, l, α , and $\Gamma(\alpha), B(\alpha, 1-\gamma)$. From (28) using lemma 1, we get the estimate

$$\begin{aligned} \|q - \tilde{q}\|_{C[0,T]} &\leq \rho_0 + (\|\varphi - \tilde{\varphi}\|_{C[0,T]} + \|\psi - \tilde{\psi}\|_{C[0,T]} \\ &\quad + \|g - \tilde{g}\|_{C^\alpha[0,T]} + \|f - \tilde{f}\|_{C(\bar{\Omega}_T)}) E_{\alpha,1} \left((\rho_1 \Gamma(\alpha))^{\frac{1}{\alpha}} t \right), \quad t \in [0, T]. \end{aligned} \quad (29)$$

This inequality implies the estimate (27), if we set $\rho = \rho_0 E_{\alpha,1} \left((\rho_1 \Gamma(\alpha))^{\frac{1}{\alpha}} T \right)$.

From Theorem 3 follows also the next assertion on uniqueness in whole for solution to the inverse problem.

Theorem 4. Let the functions φ, ψ, g, j and $\tilde{\varphi}, \tilde{\psi}, \tilde{g}, \tilde{j}$ have the same meaning as in Theorem 3 and conditions A1)-A4). Moreover, if $\varphi=\tilde{\varphi}, \psi=\tilde{\psi}, g=\tilde{g}, j=\tilde{j}$, for $t \in [0, T]$ then $q(t)=\tilde{q}(t)$.

Conclusion. In this work, an inverse problem for determining a time-dependent coefficient in a fractional diffusion-wave equation with a Caputo derivative was studied. First, the initial-boundary value problem was considered and reduced to equivalent integral equations using the Fourier method. The inverse problem was then formulated as a Volterra-type integral equation, which was solved using the contraction mapping principle. As a result, local existence and global uniqueness of the solution were established, and a stability estimate was obtained.

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A MIXED PROBLEM FOR A HIGHER-ORDER DIFFERENTIAL EQUATION WITH DEGENERACY ON THE BOUNDARY OF THE DOMAIN

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Abstract. In this paper, we investigate the well-posedness of a mixed problem for a higher-order differential equation involving a time-fractional derivative, a singular parameter $0 < \beta < 2k$, and a degeneracy with respect to the spatial variable x . By applying the method of separation of variables, a spectral problem depending on the variable x is derived, and a special operator is introduced to demonstrate the existence of eigenvalues and eigenfunctions for this spectral problem. The solution to the problem is sought in the form of a series expansion in terms of these eigenfunctions. It is shown that the operator possesses a discrete spectrum. Using the spectral theory of operators and the Mittag-Leffler function, the existence, uniqueness, and stability of the solution are proved. Furthermore, the existence and uniqueness of the solution are established for various values of the degeneracy order of the equation.

Keywords: fractional differential equation, Caputo fractional derivative, higher-order degenerate equation, mixed problem, separation of variables, spectral problem, Sturm-Liouville problem, Mittag-Leffler function, eigenvalues and eigenfunctions, weighted Sobolev space, well-posedness, existence and uniqueness, discrete spectrum, boundary degeneracy, self-adjoint operator.

СМЕШАННАЯ ЗАДАЧА ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВЫСОКОГО ПОРЯДКА С ВЫРОЖДЕНИЕМ НА ГРАНИЦЕ ОБЛАСТИ

Аннотация. В данной работе исследуется корректность постановки смешанной задачи для дифференциального уравнения высокого порядка, включающего дробную производную по времени, сингулярный параметр и вырождение по пространственной переменной. С использованием метода разделения переменных получена спектральная задача, зависящая от пространственной переменной, и введён специальный оператор для доказательства существования собственных значений и собственных функций этой задачи.

Решение задачи ищется в виде разложения в ряд по собственным функциям. Показано, что соответствующий оператор обладает дискретным спектром. С применением спектральной теории операторов и функции Миттаг-Леффлера доказаны существование, единственность и устойчивость решения. Кроме того, установлены существование и единственность решения для различных значений порядка вырождения уравнения.

Ключевые слова: дробное дифференциальное уравнение, дробная производная Капуто, вырожденное уравнение высокого порядка, смешанная задача, метод разделения переменных, спектральная задача, задача Штурма–Лиувилля, функция Миттаг-Леффлера, собственные значения и собственные функции, взвешенное пространство Соболева, корректность постановки, существование и единственность, дискретный спектр, вырождение на границе, самосопряжённый оператор.

SOHA CHEGARASIDA DEGENERATSIYAGA EGA YUQORI TARTIBLI DIFFERENSIAL TENGLAMA UCHUN ARALASH MASALA

Annotatsiya. Ushbu maqolada vaqt bo'yicha kasr tartibli hosila, singulyar parametr va fazoviy o'zgaruvchi bo'yicha degeneratsiyani o'z ichiga olgan yuqori tartibli differensial tenglama uchun aralash masalaning to'g'ri qo'yilganligi o'rganiladi. O'zgaruvchilarni ajratish usuli yordamida fazoviy o'zgaruvchiga bog'liq bo'lgan spektral masala keltirib chiqariladi hamda ushbu masala uchun xos qiymatlar va xos funksiyalar mavjudligini isbotlash maqsadida maxsus operator kiritiladi.

Masalaning yechimi xos funksiyalar bo'yicha qator ko'rinishida izlanadi. Operator diskret spektrga ega ekanligi ko'rsatiladi. Operatorlar spektral nazariyasi va Mittag-Leffler funksiyasidan foydalanib, yechimning mavjudligi, yagona ekanligi va barqarorligi isbotlanadi. Shuningdek, tenglamaning degeneratsiya tartibining turli qiymatlarida ham yechimning mavjudligi va yagonaligi aniqlanadi.

Kalit so‘zlar: kasr tartibli differensial tenglama, Kaputo hosilasi, yuqori tartibli degeneratsiyalangan tenglama, aralash masala, o‘zgaruvchilarni ajratish usuli, spektral masala, Shturm – Liuvill masalasi, Mittag–Leffler funksiyasi, xos qiymatlar va xos funksiyalar, og‘irlikli Sobolev fazosi, to‘g‘ri qo‘yilganlik, mavjudlik va yagonalik, diskret spektr, chegaraviy degeneratsiya, o‘z-o‘ziga qo‘shma operator.

Introduction. The theory of partial differential equations has extensive applications in many fields of modern science and engineering, including hydrodynamics, elasticity theory, and quantum mechanics. Currently, this theory is developing at a rapid pace in various directions. In particular, problems posed for higher-order even partial differential equations occupy a special place in this regard. An analysis of research in this direction shows that initial-boundary value problems for equations with integer-order derivatives have been extensively studied. For example, in work [1], the problem of finding a solution to the following equation in the domain $\Delta = \{(x, t) : 0 < x < p, 0 < t < T\}$ was considered:

$$\frac{\partial^{2k} u}{\partial x^{2k}} - \frac{\partial^2 u}{\partial t^2} = f(x, t) \quad (1)$$

Here, the solution must satisfy the boundary conditions

$$\frac{\partial^{2m} u}{\partial x^{2m}}(0, t) = \frac{\partial^{2m} u}{\partial x^{2m}}(p, t) = 0, m = \overline{0, k-1}, 0 \leq t \leq T, k \geq 2 \quad (2)$$

and one of the following pairs of initial conditions:

$$\begin{cases} u(x, 0) = u(x, T) = 0, 0 \leq x \leq p \\ u(x, 0) = u_t(x, T) = 0, 0 \leq x \leq p \\ u(x, 0)u(x, T), u_t(x, 0) = u_t(x, T), 0 \leq x \leq p \end{cases} \quad (3)$$

At the same time, the problem mentioned above was investigated in detail in that work. Additionally, in [1], a problem was also posed for equation (1) with given conditions and the following boundary conditions:

$$\frac{\partial^{2m+1} u}{\partial x^{2m+1}}(0, t) = \frac{\partial^{2m+1} u}{\partial x^{2m+1}}(p, t) = 0, m = \overline{0, k-1}, 0 < t < T, k \geq 2$$

In work [2], the problem for the equation in domain Δ with specific boundary conditions and initial conditions $u(x, 0) = u_t(x, 0) = 0, 0 \leq x \leq p$ was studied. In work [3], an ill-posed problem for the equation in the domain $D = \{(x, t) : 0 < x < \pi; 0 < t < 2\pi\}$ with $f(x, t) \equiv 0$ and $k = 2n+1, n \in \mathbb{N}$ was studied under the following boundary and initial conditions:

$$\frac{\partial^{2m} u}{\partial x^{2m}}(0, t) = \frac{\partial^{2m} u}{\partial x^{2m}}(\pi, t) = 0, m = \overline{0, 1, \dots, k-1}; u(\alpha\pi, t) = f(t), \quad 0 < t < 2\pi$$

In work [4], a problem for the quasilinear equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^{2k} u}{\partial x^{2k}} = f(t, x, u), k \in \mathbb{N}$$

in the domain $D_T = \{(t, x) : 0 < t < T; 0 < x < \pi\}$ was considered with the following initial conditions

$$u(0, x) = \varphi(x), \quad u_t(T, x) = \psi(x), \quad 0 < x < \pi$$

and boundary conditions

$$\left. \frac{\partial^j u}{\partial x^j} \right|_{x=0} = \left. \frac{\partial^j u}{\partial x^j} \right|_{x=\pi}, \quad j = \overline{0, 1, \dots, 2k-1}, \quad 0 < t < T$$

The Fourier method was applied to solve the problem, and the method of successive approximations in Banach space was used to prove the existence and uniqueness of the solution.

In work [5], problems for the following equation:

$$\frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^2 u}{\partial t^2} = f(x, t)$$

in domain Δ with specific conditions and $u(x, 0) = u_t(x, 0) = 0$ were investigated. The article showed that the solvability of the problem depends on whether the number k is even or odd. Modern methods of functional analysis, spectral theory, and the theory of differential equations were used in the research.

In work [6], initial-boundary value problems were studied for the following equation:

$$u_{tt} + A(x, D)u(x, t) = f(x, t)$$

where $A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is an arbitrary positive formally self-adjoint elliptic differential

operator of order $m = 2l$ with sufficiently smooth coefficients. The problem is solved in the form of a series using the classical Fourier method and the spectral problem; the existence, uniqueness, and stability of the solution were proven, and the concept of a generalized solution was introduced.

In work [7], an equation of the following form was studied in the domain $\Omega = \{(x, t) : 0 < x < 1; 0 < t < T\}$:

$$B_{\gamma-1/2}^t u + (-1)^k \frac{\partial^{2k} u}{\partial x^{2k}} = f(x, t)$$

where $k \geq 2$ is a natural number and $0 < \gamma < 1/2$ is a given real number. In this work, the problem of finding a function $u(x, t)$ satisfying the following initial conditions

$$u(x, 0) = \varphi_1(x), \lim_{t \rightarrow +0} t^{2\gamma} u_t(x, t) = \varphi_2(x)$$

and boundary conditions

$$u(0, t) = \frac{\partial u}{\partial x}(0, t) = \dots = \frac{\partial^{k-1} u}{\partial x^{k-1}}(0, t) = 0, \frac{\partial^k u}{\partial x^k}(1, t) = \frac{\partial^{k+1} u}{\partial x^{k+1}}(1, t) = \dots = \frac{\partial^{2k-1} u}{\partial x^{2k-1}}(1, t) = 0$$

was posed, and the existence and uniqueness of the solution were demonstrated using the spectral method.

In work [8], problems were considered for the equation

$$\partial_t^\rho u(x, t) + A(x, D)u(x, t) = f(x, t), x \in \Omega, t > 0$$

where ∂_t^ρ is a fractional differential operator in the sense of Riemann-Liouville of order $\rho \in (0, 1]$. $A(x, D)$ is an arbitrary positive, self-adjoint elliptic differential operator of order $m = 2l$. Ω is an arbitrary bounded domain in $\mathbb{R}^N$ with a smooth boundary. In the work, a solution satisfying the following initial condition

$$\lim_{t \rightarrow 0} \partial_t^{\rho-1} u(x, t) = \varphi(x), x \in \Omega$$

and boundary conditions

$$B_j u(x, t) = \sum_{|\alpha| \leq m_j} b_{\alpha,j}(x) D^\alpha u(x, t) = 0, 0 \leq m_j \leq m-1, j = 1, 2, \dots, l; x \in \partial\Omega$$

was sought using the Fourier method, and the existence and uniqueness of the solution were shown.

In article [9], a boundary value problem was considered in the rectangular domain $\Omega = \Omega_x \times \Omega_y$, where $\Omega_x = \{x | 0 < x < 1\}$, $\Omega_y = \{y | 0 < y < 1\}$, for the following equation involving a third-order derivative and a degenerate operator of high even order:

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} + a \frac{\partial u}{\partial x} + bu + K(y)l(u(x, y)) = f(x, y)$$

Here, $l(u)$ is a higher-order even differential operator, and $K(y)$ is a weight function that degenerates on the boundary of the domain.

$$l(u) = (-1)^s \frac{\partial^{2s} u}{\partial y^{2s}} + \frac{\partial^{s-1}}{\partial y^{s-1}} \left((-1)^{s-1} p_{s-1}(y) \frac{\partial^{s-1} u}{\partial y^{s-1}} \right) + \dots + \frac{\partial}{\partial y} \left(-p_1(y) \frac{\partial u}{\partial y} \right) + p_0(y)u$$

where $0 \leq p_j(y) \in C^j(\bar{\Omega}_y)$, $j = 0, 1, \dots, s-1$, $s \in \mathbb{N}$, $K(y) > 0$, $y \in (0, 1)$, $K(0) = K(1) = 0$.

In article [10], a Dirichlet-type problem was studied in the rectangular domain $\Omega = \{(x, y) : 0 < x < 1, -b < y < a\}$, $a, b > 0$, $a, b \in \mathbb{R}$ for the following high-order degenerate equation:

$$L[u] \equiv \begin{cases} K(x)l(u(x, y)) - D_{0y}^\alpha u(x, y) = 0, y > 0, 1 < \alpha < 2 \\ K(x)l(u(x, y)) + D_{0y}^\beta u(x, y) = 0, y < 0, 1 < \beta < 2 \end{cases}$$

where $l(u)$ is an even-order higher differential operator and $K(x)$ is a weight function that degenerates on the boundary of the domain. Mittag-Leffler functions were used in solving the fractional-order equations. The convergence of the solution series and estimates for the Green's function were found using asymptotic expansions of Mittag-Leffler functions.

In article [11], the following high-even-order degenerate partial differential equation was studied in the rectangle $\Omega = \{(x, t) : 0 < x < 1; 0 < t < T\}$:

$${}_CD_{0t}^\gamma u(x, t) - \frac{\partial^{2n+1}}{\partial x^{2n+1}} \left(x^\alpha \frac{\partial^{2n+1} u}{\partial x^{2n+1}} \right) = f(x, t)$$

where ${}_CD_{0t}^\gamma$ is the Gerasimov-Caputo differential operator of order γ , $0 < \alpha < 1$, $n \in \mathbb{N}$, $p-1 < \gamma \leq p$ and $p \leq 2$. The work aimed to find a function $u(x, t)$ satisfying the following initial and boundary conditions.

Initial conditions: $u(x, 0) = \varphi_1(x)$ for $p=1$, and $u(x, 0) = \varphi_1(x)$ and $u_t(x, 0) = \varphi_2(x)$ for $p=2$.
Boundary conditions:

$$\begin{aligned} \left. \frac{\partial^{2j} u}{\partial x^{2j}} \right|_{x=0} &= 0, \quad \left. \frac{\partial^{2j+1}}{\partial x^{2j+1}} \left(x^\alpha \frac{\partial^{2n+1} u}{\partial x^{2n+1}} \right) \right|_{x=0} = 0 \\ \left. \frac{\partial^{2j} u}{\partial x^{2j}} \right|_{x=1} &= 0, \quad \left. \frac{\partial^{2j+1}}{\partial x^{2j+1}} \left(x^\alpha \frac{\partial^{2n+1} u}{\partial x^{2n+1}} \right) \right|_{x=1} = 0 \end{aligned}$$

To solve the problem, the spectral problem was first studied and examined for self-adjointness, the Green's function for the spectral problem was constructed, and it was transformed into a Fredholm integral equation of the second kind with a symmetric kernel.

Furthermore, in works [12–16], the Cauchy problem for high-even-order equations with the Bessel operator in the upper half-plane was investigated.

Although high-order and fractional-order equations were studied separately or in certain specific cases in the aforementioned works, the issue of the influence of the degree of degeneracy on the number of boundary conditions has not been fully elucidated. In this article, unlike the research noted above, a mixed problem involving a fractional-order derivative in the sense of Caputo with respect to time and a $2k$ -order degenerate differential operator with respect to the spatial variable is investigated. The main novelty of the work is that the number of boundary conditions and their structure are strictly justified depending on the operator's degree of degeneracy β . In particular, for non-integer values of β , it is proven that the number of boundary conditions at the point $x=0$ is determined through the integer part $[\beta]$. This approach is not found in previous literature and significantly expands and complicates the scope of the problem formulation. In the article, it is proven that the defined degenerate operator for any k is positive definite, self-adjoint, and possesses a discrete spectrum.

2. Preliminaries. In this section, we present the necessary definitions and properties of fractional differential operators, the Mittag-Leffler function, and the relevant functional spaces.

2.1. Caputo fractional derivative and Mittag-Leffler functions

Definition 1 [17]. The Caputo fractional derivative of order α for a function $f(t)$ is defined as follows:

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \quad (4)$$

where $t > 0$, $m-1 < \alpha \leq m$, $m=1, 2, \dots$, and $\Gamma(z)$ is the Euler Gamma function.

Definition 2 [18]. The two-parameter Mittag-Leffler function $E_{\alpha, \beta}(t)$ is defined by the following series:

$$E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \quad |t| < \infty.$$

Lemma 1 [17]. Consider the following Cauchy problem:

$$\partial_t^\alpha u(t) - \lambda u(t) = f(t), \quad t > 0, \quad u^{(k)}(0) = u_k, \quad k=0, \dots, m-1, \quad (5)$$

where $\alpha, u_k, \lambda \in \mathbb{R}$ and $m-1 < \alpha \leq m$. Then the solution is expressed as:

$$u(t) = \sum_{k=0}^{m-1} u_k t^k E_{\alpha, k+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) f(s) ds. \quad (6)$$

Lemma 2 [18]. If $\alpha < 2$, β is an arbitrary real number, $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, and C is a real constant, then the following inequality holds:

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|} \leq C, \quad (\mu \leq |\arg(z)| \leq \pi), |z| \geq 0.$$

2.2. Functional spaces. If $t \in [0, T]$ and for each t , the function $u(\cdot, t)$ belongs to a separable Hilbert space V (for instance, $L^2(\Omega)$ or $H^1(\Omega)$), then the function u can be treated as a V -valued function depending on the variable t , denoted as $u: [0, T] \rightarrow V$.

The space $C([0, T]; V)$ is the set of continuous V -valued functions on the interval $[0, T]$, which forms a Banach space with the following norm:

$$\|u(\cdot, t)\|_{C([0, T]; V)} = \max_{0 \leq t \leq T} \|u(\cdot, t)\|_V.$$

3. Problem statement. In the domain $\Omega_T = \{(x, t): 0 < x < 1, 0 < t \leq T < \infty\}$, we consider the following fractional-order equation:

$$\partial_t^\alpha u + (-1)^k \frac{\partial^k}{\partial x^k} \left(x^\beta \frac{\partial^k u}{\partial x^k} \right) = f(x, t), \quad (7)$$

where $\partial_t^\alpha u$ is the fractional derivative in the Caputo sense, $f(x, t) \in L_2(0, 1)$ is a given function, $\alpha, \beta \in \mathbb{R}$ with $\alpha \in (1, 2)$, $0 \leq \beta < 2k$ (where β is not an integer), and T is a positive real number.

The goal is to determine a solution in the domain Ω_T that satisfies the following initial conditions:

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (8)$$

and the following boundary conditions:

$$\frac{\partial^\mu u(0, t)}{\partial x^\mu} = 0, \mu = \overline{0, k - [\beta] - 1}, \quad \frac{\partial^\mu u(1, t)}{\partial x^\mu} = 0, \mu = \overline{0, k - 1}. \quad (9)$$

Here, φ and ψ in (8) are given functions, and $[\beta]$ denotes the integer part of β .

First, let us examine the dependence of the boundary conditions on the degree of degeneracy. For this purpose, we investigate the following equation:

$$Av \equiv (-1)^k \frac{\partial^k}{\partial x^k} \left(x^\beta \frac{\partial^k v}{\partial x^k} \right) = f(x). \quad (10)$$

The domain $D(A)$ of the operator A consists of functions satisfying the following conditions:

a) $v(x) \in C^{(k-1)}[0, 1]$, and the expressions $x^\beta v^{(k)}$ are continuous on the interval $[0, 1]$ up to their $(k-1)$ -th order derivatives (inclusive). Moreover, the $(k-1)$ -th order derivative is absolutely continuous on this interval, $v^{(k-1)} \in AC[0, 1]$; b) $Av \in L_2(0, 1)$; c) The following boundary conditions are satisfied:

$$\frac{\partial^\mu v(1, t)}{\partial x^\mu} = 0, \quad \mu = \overline{0, k - 1}. \quad (11)$$

In order for the problem $\{(10), (11)\}$ to have a unique solution, we must specify the conditions at $x=0$. To achieve this, we consider the scalar product of Av and v :

$$\begin{aligned} (Av, v) &= \left((-1)^k \frac{\partial^k}{\partial x^k} \left(x^\beta \frac{\partial^k v}{\partial x^k} \right), v \right) = \left[v \frac{\partial^{k-1}}{\partial x^{k-1}} \left(x^\beta v^{(k)} \right) \right]_0^1 - \left[v_x \frac{\partial^{k-2}}{\partial x^{k-2}} \left(x^\beta v^{(k)} \right) \right]_0^1 + \left[v_{xx} \frac{\partial^{k-3}}{\partial x^{k-3}} \left(x^\beta v^{(k)} \right) \right]_0^1 - \\ &- \left[v_{xxx} \frac{\partial^{k-4}}{\partial x^{k-4}} \left(x^\beta v^{(k)} \right) \right]_0^1 + \left[v^{(4)} \frac{\partial^{k-5}}{\partial x^{k-5}} \left(x^\beta v^{(k)} \right) \right]_0^1 + \dots = \sum_{j=0}^{k-1} (-1)^{(j+k)} v^{(j)} (x^\beta v^{(k)})^{(k-1-j)} \Big|_0^1 + \int_0^1 (x^{\beta/2} v^{(k)})^2 dx \end{aligned} \quad (12)$$

Due to the boundary conditions at $x=1$, the terms associated with the upper limit vanish. However, the terms at $x=0$ must be examined separately. To this end, let us consider the first boundary condition in (9). Let us assume that $\beta > k$ and is not an integer. For such values of β , we show that the following conditions hold:

$$\lim_{x \rightarrow 0} (x^\beta v^{(k)})^{(j)} = 0, \quad (j = \overline{0, k-1}). \quad (13)$$

Now consider the case $j=1$. For sufficiently small x , the following inequality holds:

$$\lim_{x \rightarrow 0} (x^\beta v^{(k)})' = b, \text{ implying } (x^\beta v^{(k)})' > b.$$

Performing integration once yields the following inequality: $x^\beta v^{(k)} > bx$, or $v^{(k)} > bx^{1-\beta}$.

Integrating again on the interval $[0,1]$ and applying the boundary conditions, we obtain:

$$v^{(k-1)}(1) - v^{(k-1)}(0) = b \frac{1^{2-\beta}}{2-\beta} - \lim_{x \rightarrow 0} \frac{b(x^{2-\beta})}{2-\beta}.$$

Here again, a singularity arises, contradicting our assumption. When $\beta \geq k$, no boundary conditions need to be specified at $x=0$; despite the absence of such conditions, the boundary terms in (12) vanish. The proof for other values of j in (13) can be conducted in a similar manner.

Suppose $k-2 \leq \beta < k-1$. The corresponding conditions for this interval are:

$$\lim_{x \rightarrow 0} (x^\beta v^{(k)})^{(j)} \neq 0, (j = k-1, k-2), \quad \lim_{x \rightarrow 0} (x^\beta v^{(k)})^{(j)} = 0 (j = \overline{0, k-3}).$$

Based on these conclusions, we can show the validity for the general case $k-2 \leq \beta < k-1$. For these values of β , we must impose boundary conditions $v(0) = v'(0) = 0$, which ensures the boundary terms in (12) vanish at $x=0$. Continuing this process, if $k-(n+1) \leq \beta < k-n$, it is sufficient to provide the boundary conditions as follows:

$$v(0) = v'(0) = v''(0) = v'''(0) = \dots = v^{(n)}(0) = 0. \quad (14)$$

If $\beta < 1$, standard boundary conditions must be imposed at $x=0$: $v^{(\mu)}(0) = 0$, $(\mu = \overline{0, k-1})$.

If $2k-1 \leq \beta < 2k$, no conditions are required at $x=0$. From the inequality $k-(n+1) \leq \beta < k-n$, we find $n = k - [\beta] - 1$. Thus, we have proved that if boundary conditions of the form $v^{(\mu)}(0) = 0$, $(\mu = \overline{0, k - [\beta] - 1})$ are given at $x=0$, the boundary terms in (12) vanish for functions $v(x) \in D(A)$.

4. Spectral problem. We apply the method of separation of variables to the given problem. Seeking the solution to equation (7) that satisfies the boundary conditions (9) in the form $u(x, t) = T(t)v(x)$, we obtain the following Sturm-Liouville type spectral problem for $v(x)$:

$$(-1)^k \frac{d^k}{dx^k} \left(x^\beta \frac{d^k v}{dx^k} \right) = \lambda v(x) \quad (16)$$

$$v^{(\mu)}(0) = 0, \quad \mu = \overline{0, k - [\beta] - 1}; \quad v^{(\mu)}(1) = 0, \quad \mu = \overline{0, k-1} \quad (17)$$

where $T(t)$ and $v(x)$ are the unknown functions.

Our objective is to prove the existence of eigenvalues and eigenfunctions for the problem $\{(16), (17)\}$.

To this end, we consider the operator: $Av = (-1)^k \frac{d^k}{dx^k} \left(x^\beta \frac{d^k v}{dx^k} \right)$

The domain $D(A)$ consists of functions $v(x)$ possessing the following properties:

- a) $v(x) \in C^{(k-1)}[0,1]$; b) $x^\beta v^{(k)} \in C[0,1]$, $(x^\beta v^{(k)})^{(k-1)} \in AC[0,1]$, and $Av \in L_2(0,1)$;
- c) $v^{(\mu)}(0) = 0, \mu = \overline{0, k - [\beta] - 1}; v^{(\mu)}(1) = 0, \mu = \overline{0, k-1}$.

Let $\dot{W}_{2,\beta}^k$ denote the closure of the set $D(A)$ with respect to the norm:

$$\|v\|_{\dot{W}_{2,\beta}^k}^2 = \int_0^1 [v^2(x) + x^\beta (v^{(k)}(x))^2] dx$$

Theorem 1. Let $0 < \beta \leq 2k$ and β not be an integer. The operator A defined in $L_2(0,1)$ is positively defined.

Proof. First, we show that the operator A is symmetric. Indeed, the domain $D(A)$ contains a set of functions that are compact on the interval $[0,1]$ and dense in $L_2(0,1)$. Consequently, the set $D(A)$ is inherently dense in $L_2(0,1)$. Now, for functions $v, w \in D(A)$, we construct the scalar product:

$$(Av, w) = \left((-1)^k (x^\beta v^{(k)})^{(k)}, w \right) = (-1)^k \int_0^1 (x^\beta v^{(k)})^{(k)} w dx$$

Integrating this scalar product by parts repeatedly, we obtain:

$$\int_0^1 (-1)^k \left((x^\beta v^{(k)})^{(k)} \right) w dx = \sum_{j=0}^{k-1} (-1)^{(j+k)} w^{(j)} (x^\beta v^{(k)})^{(k-1-j)} \Big|_0^1 + \int_0^1 (x^\beta v^{(k)}) w^{(k)} dx. \quad (18)$$

By verifying the general formula and utilizing the boundary conditions imposed on $w(x)$ (which are identical to those for $v(x)$), we conclude that the operator A is symmetric: $(Av, w) = (v, Aw)$

Next, we show that for any $v \in D(A)$, there exists a positive constant $\gamma > 0$ such that the operator A satisfies the inequality:

$$(Av, v) \geq \gamma^2 \|v\|_{L_2}^2 \quad (19)$$

Setting $w = v$ in (18), we obtain:

$$(Av, v) = \int_0^1 x^\beta (v^{(k)})^2 dx \quad (20)$$

It can be shown that the following equality holds: $v(x) = \frac{1}{(k-1)!} \int_x^1 (x-t)^{(k-1)} v^{(k)}(t) dt$

By squaring $v(x)$ and integrating over $[0, 1]$, and then applying the Cauchy-Schwarz inequality, we derive the following estimate:

$$\int_0^1 v^2(x) dx \leq \frac{1}{[(k-1)!]^2} \int_0^1 \left(\int_x^1 \frac{(x-t)^{2(k-1)}}{t^\beta} dt \right) dx \left(\int_0^1 t^\beta [v^{(k)}(t)]^2 dt \right) \quad (21)$$

We introduce the notation: $\gamma^{-2} = \frac{1}{[(k-1)!]^2} \int_0^1 \left(\int_x^1 \frac{(x-t)^{2(k-1)}}{t^\beta} dt \right) dx$

where γ is a constant depending on k and β . Thus, for $\beta < 2k$, the inequality (19) is proved. This completes the proof of Theorem 1.

Theorem 2. Let $0 < \beta \leq 2k$ and β not be an integer. Then the operator A has a discrete spectrum.

Proof. Given that the operator A is positively defined, we define the norm of the energy space

$H_A(0,1)$ as: $\|v\|_A^2 = \int_0^1 \sum_{\nu=0}^k x^\beta [v^{(\nu)}]^2 dx$

It can be shown that $H_A(0,1)$ coincides with the space $\dot{W}_{2,\beta}^k$ as a set of functions, and their norms are equivalent. Suppose M is a set of functions bounded in the energy space H_A , i.e., $\|v\|_{H_A} \leq c$. From (19)

and (20), it follows that: $\|v\|_{W_{2,\beta}^k}^2 = \int_0^1 [v^2(x) + x^\beta (v^{(k)}(x))^2] dx \leq \text{const}$

According to the weighted class theorems provided in [19], the set M is compact in the following spaces:

a) If $0 \leq \beta < 2k-1$, M is compact in the space of continuous functions.

b) If $\beta = 2k-1$, M is compact in L_{q^*} for any $q^* > 1$.

c) If $\beta > 2k-1$, M is compact in L_q provided $q < \frac{2}{\beta+1-2k}$.

It follows that if $0 \leq \beta < 2k$, then M is compact in $L_2[0,1]$. Based on Theorem 3 in Section 40 of [20], we conclude that the spectrum of operator A is discrete. The eigenfunctions of operator A possess the following properties:

• They are orthonormal in L_2 : $\int_0^1 v_i v_j dx = \delta_{ij}$ (22)

• They are orthogonal in H_A : $(v_i, v_j)_A = \int_0^1 \sum_{k=0}^n x^\beta v_i^{(k)} v_j^{(k)} dx = \lambda_i \delta_{ij}$ (23)

where $\{v_i(x)\}_{i=1}^\infty$ and $\{\lambda_i\}_{i=1}^\infty$ are the eigenfunctions and eigenvalues of operator A , respectively.

5. Order of Fourier coefficients. In this section, we investigate the convergence of certain series that will be required throughout the paper.

Lemma 3. Let $f \in \dot{W}_{2,\beta}^{(k)}(0,1)$. Then the following Bessel-type inequality holds:

$$\sum_{n=1}^{+\infty} \lambda_n f_n^2, \int_0^1 x^\beta \left[f^{(k)}(x) \right]^2 dx \quad (24)$$

From this, it follows that the numerical series on the left side of (24) is convergent. Here, $f_n = \int_0^1 f(x) v_n(x) dx$ are the Fourier coefficients of the function $f(x)$ with respect to the system of eigenfunctions $\{v_n\}_{n=1}^{+\infty}$, and $\lambda_n (n=1, 2, 3, \dots)$ are the eigenvalues of the spectral problem $\{(16), (17)\}$.

Proof. According to the definition of generalized eigenfunctions, for any function $f(x) \in W_{2,\beta}^{(k)}(0,1)$, the following identity holds:

$$\int_0^1 x^\beta v_n^{(k)}(x) f^{(k)}(x) dx = \lambda_n f_n \quad (25)$$

Let us consider the following non-negative expression: $\int_0^1 x^\beta \left(f^{(k)}(x) - \sum_{i=1}^n f_i v_i^{(k)}(x) \right)^2 dx \geq 0$

Expanding the square, we obtain:

$$\int_0^1 x^\beta (f^{(k)}(x))^2 dx - 2 \sum_{i=1}^n f_i \int_0^1 x^\beta f^{(k)}(x) v_i^{(k)}(x) dx + \sum_{i=1}^n f_i^2 \int_0^1 x^\beta (v_i^{(k)}(x))^2 dx + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n f_i f_j \int_0^1 x^\beta v_i^{(k)}(x) v_j^{(k)}(x) dx \geq 0$$

Here, the last term involving the double sum vanishes due to the orthogonality condition (23). Taking into account the identities (22), (23), and (25), we have: $\sum_{i=1}^n \lambda_i f_i^2 \leq \int_0^1 x^\beta \left[f^{(k)}(x) \right]^2 dx$

By passing to the limit as $n \rightarrow +\infty$, we arrive at the inequality (24).

Now, we introduce the following notation for the powers of the operator: $A^m f = \underbrace{A(A(\dots(A f) \dots))}_{m \text{ times}}$

Lemma 4. Suppose that the following conditions are satisfied:

1. For any even m , $f, Af, \dots, A^{\frac{m}{2}} f \in \dot{W}_{2,\beta}^{(k)}(0,1)$;
2. For any odd m , $f, Af, \dots, A^{\frac{m-1}{2}} f \in \dot{W}_{2,\beta}^{(k)}(0,1)$ and $A^{\frac{m+1}{2}} f \in L^2(0,1)$.

Then, for any $m \in \mathbb{N}$, the following Bessel-type inequalities hold:

$$\sum_{n=1}^{+\infty} \lambda_n^{m+1} f_n^2 \leq \begin{cases} \int_0^1 x^\beta \left[\left(A^{\frac{m}{2}} f \right)^{(k)} \right]^2 dx, & \text{for even } m, \\ \int_0^1 \left[A^{\frac{m+1}{2}} f \right]^2 dx, & \text{for odd } m \end{cases} \quad (26)$$

Proof. Let m be an even natural number. By applying the integration by parts rule repeatedly, we obtain from (24) the following relation:

$$(Af, v_n) = \lambda_n f_n = \lambda_n (f, v_n) \quad (27)$$

Replacing f with Af in (27), we get: $(A^2 f, v_n) = \lambda_n (Af, v_n) = \lambda_n^2 f_n$. Continuing this process, it can be shown that: $(A^p f, v_n) = \lambda_n^p f_n$. This implies that the n -th Fourier coefficient of the function $A^p f$ is $\lambda_n^p f_n$.

a) Since m is even, let $m = 2p$ where p is an integer. By the hypothesis, $A^{m/2} f \in \dot{W}_{2,\beta}^{(k)}(0,1)$. Then:

$$(A^{\frac{m}{2}} f, v_n) = \lambda_n^{\frac{m}{2}} f_n \quad (28)$$

Equality (28) shows that $\lambda_n^{m/2} f_n$ is the Fourier coefficient of the function $A^{m/2} f$ with respect to the system of eigenfunctions. Applying Lemma 3 to $g = A^{m/2} f$, we obtain: $\sum_{n=1}^{+\infty} \lambda_n g_n^2, \int_0^1 x^\beta \left[g^{(k)}(x) \right]^2 dx$

Substituting $g_n = \lambda_n^{m/2} f_n$, we get: $\sum_{n=1}^{+\infty} \lambda_n \left(\lambda_n^{\frac{m}{2}} f_n \right)^2 \gg \int_0^1 x^\beta \left[\left(A^{\frac{m}{2}} f \right)^{(k)} \right]^2 dx$

Thus, Lemma 4 is proved for even m .

Now consider the case where m is odd. In this case, from (28) we have: $(A^{\frac{m+1}{2}} f, v_n) = \lambda_n^{\frac{m+1}{2}} f_n$

Assuming $A^{\frac{m+1}{2}} f \in L^2(0,1)$ and applying the classical Bessel inequality for an orthonormal system $\{v_n\}$, we obtain: $\sum_{n=1}^{+\infty} \left(A^{\frac{m+1}{2}} f, v_n \right)^2 = \sum_{n=1}^{+\infty} \lambda_n^{m+1} f_n^2 \gg \int_0^1 \left[A^{\frac{m+1}{2}} f \right]^2 dx$

This completes the proof of Lemma 4.

Lemma 5. Suppose that $f(x,t) \in C([0,T], \dot{W}_{2,\beta}^{(k)}(0,1))$ and $Af(x,t) \in C([0,T], L^2(0,1))$. Then the following series $\sum_{n=1}^{+\infty} \lambda_n^2 f_n^2(t)$ converges absolutely and uniformly on the interval $[0,T]$.

Proof. Our goal is to demonstrate the absolute and uniform convergence of the series $\sum_{n=1}^{+\infty} \lambda_n^2 f_n^2(t)$ with respect to t on the interval $[0,T]$.

Fourier Expansion: As previously established, $v_n(x)$ are the eigenfunctions of the operator A , and λ_n are the corresponding eigenvalues. Thus, the function $f(x,t)$ can be expanded into a Fourier series:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) v_n(x), \quad f_n(t) = \int_0^1 f(x,t) v_n(x) dx.$$

Now, we apply the operator A to the function $f(x,t)$. Since the operator is linear, it acts on each term of the series: $Af(x,t) = A \left(\sum_{n=1}^{\infty} f_n(t) v_n(x) \right) = \sum_{n=1}^{\infty} f_n(t) (Av_n(x))$. Taking into account that $Av_n(x) = \lambda_n v_n(x)$, we obtain: $Af(x,t) = \sum_{n=1}^{\infty} \lambda_n f_n(t) v_n(x)$.

Consequently, the Fourier coefficients of the function $Af(x,t)$ are exactly $\lambda_n f_n(t)$. Parseval's identity establishes the relationship between the norm of the function and its coefficients:

$$\|Af(x,t)\|_{L_2}^2 = \int_0^1 [Af(x,t)]^2 dx = \sum_{n=1}^{\infty} (\lambda_n f_n(t))^2.$$

Since $Af(x,t) \in C([0,T], L^2(0,1))$, by Dini's theorem, Lemma 5 is proved.

Lemma 6. Suppose that $f(x,t), Af(x,t) \in C([0,T], \dot{W}_{2,\beta}^{(k)}(0,1))$. Then the following series $\sum_{n=1}^{+\infty} \lambda_n^3 f_n^2(t)$ converges absolutely and uniformly on the interval $[0,T]$.

Proof. We utilize the fact that $f_n(t)$ is the Fourier coefficient of $f(x,t)$ with respect to the eigenfunctions $v_n(x)$: $f_n(t) = (f, v_n)$. According to Lemma 4, the Fourier coefficient of the function Af is given by: $(Af, v_n) = \lambda_n f_n(t)$. By the hypothesis, the function $g(x,t) = Af(x,t)$ belongs to the space $\dot{W}_{2,\beta}^{(k)}(0,1)$ for each fixed $t \in [0,T]$. We apply Lemma 3 (the Bessel-type inequality) to the function $g(x,t)$:

$$\sum_{n=1}^{+\infty} \lambda_n g_n^2(t) \leq \int_0^1 x^\beta \left[\frac{\partial^k}{\partial x^k} g(x,t) \right]^2 dx.$$

Substituting $g_n(t) = \lambda_n f_n(t)$, we have: $\sum_{n=1}^{+\infty} \lambda_n (\lambda_n f_n(t))^2 = \sum_{n=1}^{+\infty} \lambda_n^3 f_n^2(t) \leq \int_0^1 x^\beta [(Af)^{(k)}]^2 dx$.

The condition $Af \in C([0, T], \dot{W}_{2,\beta}^{(k)}(0, 1))$ implies that the integral on the right-hand side is a continuous function of t . Specifically, it is bounded on the closed interval $[0, T]$: $\int_0^1 x^\beta [(Af)^{(k)}]^2 dx < \infty, \quad \forall t \in [0, T]$.

Now we demonstrate uniform convergence. The series $\sum \lambda_n^3 f_n^2(t)$ is a series with positive terms. For each fixed t , the series converges. It is known that if the partial sums $S_N(t)$ of a functional series with positive terms are continuous and converge pointwise to a continuous function, then by Dini's theorem, the convergence is uniform. In this case, $S_N(t) = \sum_{n=1}^N \lambda_n^3 f_n^2(t)$ is continuous as a sum of continuous functions.

Since the norm of Af in the space $\dot{W}_{2,\beta}^{(k)}$ is continuous, the sum of the series is also continuous. According to Dini's theorem, Lemma 6 is proved.

6. Existence and uniqueness of the solution. We prove the existence and uniqueness of the solution to the problem. To do this, we consider two cases:

Case 1: $0 < \beta < 2k$ and β is not an integer. Let $\{v_k\}_{k=1}^{+\infty}$ and $\{\lambda_k\}_{k=1}^{+\infty}$ be the eigenfunctions and eigenvalues of the spectral problem $\{(16), (17)\}$, respectively. We seek the solution to the problem in the following form:

$$u(x, t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x) \quad (29)$$

where $u_k(t) = (u, v_k)_{L^2(0,1)}$ for $k \in \mathbb{N}$. Substituting (29) into (7) and introducing $f_k(t) = (f, v_k)_{L^2(0,1)}$, we obtain from (7):

$$\partial_t^\alpha u_k(t) + \lambda_k u_k(t) = f_k(t), \quad k \in \mathbb{N}, \quad t \in (0, T) \quad (30)$$

Furthermore, from the initial conditions (8), we have:

$$u_k(0) = \varphi_k, \quad u'_k(0) = \psi_k, \quad k \in \mathbb{N} \quad (31)$$

where φ_k and ψ_k are the Fourier coefficients of the functions $\varphi(x)$ and $\psi(x)$, respectively, with respect to the system of eigenfunctions $\{v_k\}_{k=1}^{+\infty}$: $\varphi_k = \int_0^1 \varphi(x) v_k(x) dx$, $\psi_k = \int_0^1 \psi(x) v_k(x) dx$.

It is easy to verify that the solution to equation (30) satisfying the conditions (31) is given by:

$$u_k(t) = \varphi_k E_{\alpha,1}(-\lambda_k t^\alpha) + \psi_k t E_{\alpha,2}(-\lambda_k t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[-\lambda_k(t-s)^\alpha] f_k(s) ds$$

Substituting this expression of $u_k(t)$ into (29), we have

$$u(x, t) = \sum_{k=1}^{+\infty} \left[\varphi_k E_{\alpha,1}(-\lambda_k t^\alpha) + \psi_k t E_{\alpha,2}(-\lambda_k t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[-\lambda_k(t-s)^\alpha] f_k(s) ds \right] v_k(x). \quad (32)$$

Theorem 3. Suppose that $0 < \beta < 2k-1$ and β is not an integer. If $\varphi, A\varphi \in \dot{W}_{2,\beta}^{(k)}(0, 1)$, $\psi, A\psi \in \dot{W}_{2,\beta}^{(k)}(0, 1)$, and $f, Af \in C([0, T], \dot{W}_{2,\beta}^{(k)}(0, 1))$, then the function $u(x, t)$ defined by (32) is the classical solution to the problem.

Proof. To prove Theorem 3, it is sufficient to show the continuity of $\partial_t^\alpha u$ and Au in the domain $\bar{\Omega}_T$. Differentiating (32), we have:

$$Au \equiv (-1)^k \frac{\partial^k}{\partial x^k} \left(x^\beta \frac{\partial^k u}{\partial x^k} \right) = \sum_{k=1}^{+\infty} \lambda_k u_k(t) v_k(x) \quad (33)$$

Taking the norm in H_A : $\|Au\|_{H_A}^2 = \sum_{k=1}^{+\infty} \lambda_k^3 u_k^2(t)$

Using the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ and Lemma 2:

$$\|Au\|_{H_A}^2 \leq C_1 \sum_{k=1}^{+\infty} \lambda_k^3 \varphi_k^2 + C_2 \sum_{k=1}^{+\infty} \lambda_k^3 \psi_k^2 + C_3 \sum_{k=1}^{+\infty} \lambda_k^3 \int_0^T f_k^2(s) ds$$

The convergence follows from Lemma 4 and Lemma 6. Thus, $\|Au\|_{H_A}^2 \leq \text{const}$, which implies $Au \in C[0,1]$ by Kondrashov's embedding theorems. The continuity with respect to t follows from the uniform convergence of the series. Thus, $Au \in C(\bar{\Omega}_T)$.

Theorem 4. Let $2k-1 < \beta < 2k$. If $\varphi \in \dot{W}_{2,\beta}^{(k)}(0,1)$, $A\varphi \in L_2(0,1)$, $\psi \in \dot{W}_{2,\beta}^{(k)}(0,1)$, $A\psi \in L_2(0,1)$, and f, Af satisfy the corresponding continuity conditions, then $u(x,t)$ is a solution such that $Au, \partial_t^\alpha u \in C([0,T], L_2(0,1))$.

Proof. Similar to Theorem 3, we estimate $\|Au\|_{L^2}^2 = \sum \lambda_k^2 u_k^2(t)$. Based on Lemma 4 and Lemma 5, the series converges uniformly on $[0,T]$ by the Weierstrass M-test. The uniqueness of the solution follows from the completeness of the system $\{v_k(x)\}$ in $L_2(0,1)$.

7. Conclusion. This paper is devoted to studying the well-posedness of a mixed problem for a higher-order time-fractional differential equation with space-variable degeneracy. Our approach demonstrates that the formulation of the boundary conditions for the space-degenerate equation depends fundamentally on the degree of degeneration β . It was established that as the order of degeneracy increases within the interval $0 \leq \beta < 2k$, the number of required boundary conditions at the singular line $x=0$ decreases.

Additionally, the degree of degeneration affects the qualitative form and smoothness of the solution to the considered problem. We proved that if the degree of degeneration satisfies $0 < \beta < 2k-1$, the solution is a classical solution, characterized by the continuity of the fractional derivative and the operator A in the closed domain. In the case where $2k-1 < \beta < 2k$, the solution is shown to be a less smooth solution belonging to the $L_2(0,1)$ space with respect to the spatial variable. These results were obtained by utilizing the method of separation of variables, the spectral theory of operators, and the properties of the Mittag-Leffler function.

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POSITIVE DEFINITENESS OF THE EXPONENTIAL MATRIX FOR BINARY INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the exponential matrix corresponding to binary infinite system of differential equations is studied. A new equality expressing the positive definiteness of the obtained exponential matrix is proved. The proof of this equality is based on the main properties of the exponential matrix as well as the method of mathematical induction. The obtained result can be applied in the study of optimal control and differential games associated with infinite systems.

Key words: exponential matrix, positive definiteness, binary infinite systems of differential equations, infinite-dimensional systems, vector, mathematic induction method, expansion of series.

IKKITALIK CHEKSIZ DIFFERENSIAL TENGLAMALAR SISTEMALARI UCHUN EKSPONENTSIAL MATRITSANING MUSBAT ANIQLANGANLIGI

Annotatsiya. Ushbu maqolada ikkitalik cheksiz differensial tenglamalar sistemalariga mos keladigan eksponentsial matritsa tahlil qilinadi. Hosil qilingan eksponentsial matritsaning musbat aniqlanganligini ifodalovchi yangi tenglik isbotlanadi. Ushbu tenglikni isbotlash jarayonida eksponentsial matritsaning muhim xossalardan hamda matematik induksiya usulidan foydalaniladi. Olingan natija cheksiz sistemalar bilan bog'liq optimal boshqaruv hamda differensial o'yinlar masalalarini o'rganishda qo'llanishi mumkin.

Kalit so'zlar: Eksponentsial matritsa, musbat aniqlanganlik, ikkitalik cheksiz differensial tenglamalar sistemalari, cheksiz o'lchamli sistemalar, vektor, matematik induksiya metodi, qatorga yoyish.

ПОЛОЖИТЕЛЬНАЯ ОПРЕДЕЛЁННОСТЬ ЭКСПОНЕНЦИАЛЬНОЙ МАТРИЦЫ ДЛЯ ДВОЙНЫХ БЕСКОНЕЧНЫХ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Аннотация. В данной статье исследуется экспоненциальная матрица, соответствующая двумерной бесконечной системе дифференциальных уравнений. Доказано новое равенство, выражающее положительную определённость полученной экспоненциальной матрицы. При доказательстве данного равенства использованы основные свойства экспоненциальной матрицы, а также метод математической индукции. Полученный результат может быть применён при исследовании задач оптимального управления и дифференциальных игр, связанных с бесконечными системами.

Ключевые слова: матрица экспоненты, положительная определённость, бинарные бесконечные системы дифференциальных уравнений, бесконечномерные системы, вектор, метод математической индукции, разложение ряда.

Introduction. The exponential matrix is one of the fundamental notions in the course of higher algebra. The exponential matrix and many related statements are extensively applied in various areas of mathematics. In particular, it has been widely used in numerous scientific studies in fields such as differential equations, dynamical systems, differential games, and control theory. In many research papers, it can be observed that the application of the properties of the exponential matrix significantly simplifies the analysis. In this paper, several statements involving the exponential matrix are presented. These statements are proved by means of the method of mathematical induction. In addition, the following properties of the exponential matrix are widely used throughout this work.

Literature review and methodology. The theory of exponential matrices is one of the fundamental topics in linear algebra and functional analysis and plays an important role in the study of systems of differential equations, dynamical systems, optimal control, and differential games. A comprehensive theoretical foundation of matrix functions, including the matrix exponential and its positive definite properties, is presented in the classical monograph by R. Bhatia (*Matrix Analysis*), where spectral properties and positivity issues are investigated in detail,[1]

From a numerical and applied point of view, A. Householder studied matrix exponentials in connection with linear operators and systems of differential equations. His results provide essential tools for extending finite-dimensional concepts to infinite-dimensional spaces, which is crucial in the analysis of infinite systems,[2]

Infinite systems of differential equations have been extensively investigated in the works of Kh. Sh. Qushaqov. In particular, existence and uniqueness theorems for binary and ternary infinite systems were established. These studies highlight the importance of infinite matrices and operators in Hilbert spaces and motivate further investigation of their exponential representations,[3]

Additional insight into the structure of exponential matrices is given in the work of D. Bernstein, where the relationship between orthogonal matrices and matrix exponentials is discussed. Such results are closely related to the study of positivity and definiteness properties of exponential matrices,[7]

However, despite the wide range of studies devoted to matrix exponentials and infinite systems of differential equations, the problem of expressing and proving the positive definiteness of an exponential matrix corresponding to a binary infinite system through an explicit equality has not been sufficiently addressed in the literature. The present paper aims to fill this gap by proving a new equality that characterizes the positive definite property of the resulting exponential matrix using its intrinsic properties.

Results and discussion. We consider the following infinite system of differential equation in Hilbert space l_2 :

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + y_i + w_{i1}(t), & x_i(0) &= x_{i0}, \\ \dot{y}_i &= -\lambda_i y_i + z_i + w_{i2}(t), & y_i(0) &= y_{i0}, \quad i = 1, 2, \dots, \end{aligned} \quad (1)$$

where λ_i is a given non negative real number, and

$$x_0 = (x_{10}, x_{20}, \dots), \quad y_0 = (y_{10}, y_{20}, \dots) \in l_2.$$

From the coefficients of this system, the following matrix can be obtained:

$$A_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}, \quad i = 1, 2, \dots, \quad t \in [0; T]$$

Based on this matrix, we examine the following property of the exponential matrix.

Property. If

$$A_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}, \quad i = 1, 2, \dots, \quad t \in [0; T],$$

then the following statement is true;

$$e^{A_i t} = e^{-\lambda_i t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad i = 1, 2, \dots \quad (2)$$

Let's say $W(t)$ is equal to

$$W(t) = \int_0^t e^{-As} \cdot e^{-A^T s} ds.$$

Here

$$A = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}, \quad \lambda > 0, \quad e^{At} = e^{-\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Then

$$W(t) = \int_0^t \begin{bmatrix} e^{\lambda s} & -s e^{\lambda s} \\ 0 & e^{\lambda s} \end{bmatrix} \begin{bmatrix} e^{\lambda s} & 0 \\ -s e^{\lambda s} & e^{\lambda s} \end{bmatrix} ds = \begin{bmatrix} \int_0^t e^{2\lambda s} (1 + s^2) ds & \int_0^t e^{2\lambda s} (-s) ds \\ \int_0^t e^{2\lambda s} (-s) ds & \int_0^t e^{2\lambda s} ds \end{bmatrix}$$

In the last expression we introduced the notation as follows

$$\begin{aligned} \varphi_{11}(t) &= \int_0^t e^{2\lambda s} (1 + s^2) ds, \quad \varphi_{12}(t) = \varphi_{21}(t) = \int_0^t e^{2\lambda s} (-s) ds, \\ \varphi_{22}(t) &= \int_0^t e^{2\lambda s} ds, \quad \lambda > 0. \end{aligned} \quad (3)$$

Proof of Property. To prove equality (2) in the Property, we use the following expansion of the exponential function $f(x) = e^x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (4)$$

By denoting $x = A_i t$, we obtain the following:

$$e^{A_i t} = E_3 + A_i t + \frac{(A_i t)^2}{2!} + \frac{(A_i t)^3}{3!} + \dots + \frac{(A_i t)^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{(A_i t)^n}{n!} \quad (5)$$

For convenience, we determine A_i^n (for $n = 2, 3, \dots$) separately in equality (4):

$$\begin{aligned} A_i^2 &= \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} \cdot \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i^2 & -2\lambda_i \\ 0 & \lambda_i^2 \end{bmatrix}, \\ A_i^3 &= \begin{bmatrix} \lambda_i^2 & -2\lambda_i \\ 0 & \lambda_i^2 \end{bmatrix} \cdot \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} = \begin{bmatrix} -\lambda_i^3 & 3\lambda_i^2 \\ 0 & -\lambda_i^3 \end{bmatrix}, \\ A_i^4 &= \begin{bmatrix} -\lambda_i^3 & 3\lambda_i^2 \\ 0 & -\lambda_i^3 \end{bmatrix} \cdot \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i^4 & -4\lambda_i^3 \\ 0 & \lambda_i^4 \end{bmatrix}. \end{aligned}$$

Through the continuation of this procedure, we derive:

$$A_i^n = \begin{bmatrix} (-1)^n \lambda_i^n & (-1)^{n-1} \cdot n \cdot \lambda_i^{n-1} \\ 0 & (-1)^n \lambda_i^n \end{bmatrix} \quad (6)$$

Now, we demonstrate that equality (5), obtained for A_i^n using the inductive method, holds for any $n \in \mathbb{N}$ by means of mathematical induction; in other words, we prove that:

$$\begin{aligned} A_i^{n+1} &= \begin{bmatrix} (-1)^{n+1} \lambda_i^{n+1} & (-1)^n \cdot (n+1) \cdot \lambda_i^n \\ 0 & (-1)^{n+1} \lambda_i^{n+1} \end{bmatrix} \\ &= A_i^{n+1} = A_i^n \cdot A_i = \\ &= \begin{bmatrix} (-1)^n \lambda_i^n & (-1)^{n-1} \cdot n \cdot \lambda_i^{n-1} \\ 0 & (-1)^n \lambda_i^n \end{bmatrix} \cdot \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} = \\ &= \begin{bmatrix} (-1)^{n+1} \lambda_i^{n+1} & (-1)^n \cdot (n+1) \cdot \lambda_i^n \\ 0 & (-1)^{n+1} \lambda_i^{n+1} \end{bmatrix} \end{aligned}$$

This shows that (5) equality is true for any $n \in \mathbb{N}$.

Thus,

$$\begin{aligned} e^{A_i t} &= \sum_{n=0}^{\infty} \frac{(A_i \cdot t)^n}{n!} = \sum_{n=0}^{\infty} \frac{A_i^n \cdot t^n}{n!} = \\ &= \sum_{n=0}^{\infty} \begin{bmatrix} \frac{(-\lambda_i t)^n}{n!} & \frac{(-\lambda_i t)^{n-1}}{(n-1)!} \cdot t \\ 0 & \frac{(-\lambda_i t)^n}{n!} \end{bmatrix} = e^{-\lambda_i} \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The property was proved.

Theorem. If $\eta_0 = (x_0; y_0)$, $\eta_0 \neq 0$ then the following statements are true:

$$\eta_0 \omega_i(t) \eta_0^* = \int_0^t |e^{-A^* s} \eta_0^*|^2 ds \quad (7)$$

Proof. It can be readily shown that

$$e^{-A^* t} = e^{\lambda t} \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

This will help to motivate us to continue our work on $\omega_i(t)$. Let us now prove problem (7). First, we expand the left-hand side of (7). We have $\omega_i(t)$ and a nonzero vector with constant components,

$\eta_0 = (x_0; y_0)$, where at least one element of η_0 is nonzero.

$$\begin{aligned} \eta_0 \omega_i(t) \eta_0^* &= (x_0; y_0) \begin{pmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \\ &= (x_0 \varphi_{11}(t) + y_0 \varphi_{21}(t); x_0 \varphi_{12}(t) + y_0 \varphi_{22}(t)) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \\ &= x_0^2 \varphi_{11}(t) + x_0 y_0 \varphi_{21}(t) + x_0 y_0 \varphi_{12}(t) + y_0^2 \varphi_{22}(t) = x_0^2 \varphi_{11}(t) + 2x_0 y_0 \varphi_{21}(t) + y_0^2 \varphi_{22}(t) \end{aligned}$$

At this stage, we incorporate the notations from (3) into the associated expression:

$$\int_0^t (x_0^2(1+s^2) + 2x_0y_0(-s) + y_0^2)e^{2\lambda s} ds = \int_0^t e^{2\lambda s} (x_0^2 + (x_0^2s^2 - 2x_0y_0s + y_0^2)) ds$$

It is easy to see that the above expressions are expansions of the complete quadratic formula.

$$\int_0^t e^{2\lambda s} (x_0^2 + (x_0s - y_0)^2) ds = \int_0^t \left[(x_0 \cdot e^{\lambda s})^2 + (x_0se^{\lambda s} - y_0e^{\lambda s})^2 \right] ds$$

There we can easily see that

$$\begin{aligned} & (x_0 \cdot e^{\lambda s})^2 + (x_0se^{\lambda s} - y_0e^{\lambda s})^2 = \\ & = |x_0 \cdot e^{\lambda s}; -x_0se^{\lambda s} + y_0e^{\lambda s}|^2 = \\ & = \left| \begin{pmatrix} e^{\lambda s} & 0 \\ -se^{\lambda s} & e^{\lambda s} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right|^2 = |e^{-A^*s} \eta_0^*|^2 \end{aligned}$$

Thus,

$$\eta_0 \omega_i(t) \eta_0^* = \int_0^t |e^{-A^*s} \eta_0^*|^2 ds.$$

Thus, the proof is established.

Conclusion. In this paper, we investigate the positive definiteness of the exponential matrix corresponding to binary infinite systems of differential equations. By utilizing the series expansion of the matrix exponential along with the method of mathematical induction, a new equality describing its positive definite property has been established. The resulting findings hold practical significance for the analysis of infinite-dimensional systems, especially in applications related to optimal control and differential game theory. The proposed methodology lays a theoretical foundation for further exploration of the properties and uses of exponential matrices linked to infinite systems.

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PTOLEMEY TEOREMASI VA UNING BA‘ZI TATBIQLARI

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Annotatsiya. Ushbu maqola klassik planimetriyaning fundamental natijalaridan biri Ptolemey teoremasining trigonometrik tabiatini tadqiq etishga bag‘ishlanadi. Maqolada teoremaning an‘anaviy geometrik tushunchadan tashqariga chiqib, trigonometrik ayniyatlarni isbotlashdagi o‘rni va vatar uzunliklari orqali isbotlanish mexanizmi tahlil qilinadi. Shuningdek, maqola doirasida teoremaning matematika olimpiadalarida nostandart geometrik muammolarni yechishdagi samaradorligi konkret misollar yordamida asoslab beriladi. Tadqiqot natijalari geometriya va trigonometriyaning uzviy bog‘liqligini ko‘rsatib, o‘quvchida matematik qonuniyatlarga nisbatan yaxlit tasavvurni shakllantirishga xizmat qiladi.

Kalit so‘zlar: Ptolemey teoremasi, vatar va yoy munosabatlari, sinuslar teoremasi, trigonometrik ayniyat, siklik (aylanaga ichki chizilgan) to‘rtburchaklar, planimetrik modellash, masala va yechim.

PTOLEMY’S THEOREM AND SOME OF ITS APPLICATIONS

Abstract. This article is devoted to the study of the trigonometric nature of Ptolemy’s theorem, one of the fundamental results of classical plane geometry. The paper analyzes the role of the theorem beyond its traditional geometric interpretation, particularly in proving trigonometric identities and explaining the mechanism of its proof through chord lengths. Furthermore, within the scope of the article, the effectiveness of the theorem in solving non-standard geometric problems in mathematics olympiads is substantiated with concrete examples. The research findings demonstrate the intrinsic connection between geometry and trigonometry and contribute to forming a holistic understanding of mathematical laws.

Keywords: Ptolemy’s theorem, chord and arc relations, law of sines, trigonometric identity, cyclic (inscribed) quadrilaterals, planimetric modeling, problem and solution.

ТЕОРЕМА ПТОЛЕМЕЯ И НЕКОТОРЫЕ ЕЁ ПРИМЕНЕНИЯ

Аннотация. Данная статья посвящена исследованию тригонометрической природы теоремы Птолемея, одного из фундаментальных результатов классической планиметрии. В работе анализируется значение теоремы за пределами её традиционного геометрического понимания, в частности её роль в доказательстве тригонометрических тождеств и механизм доказательства через длины хорд. Кроме того, в рамках статьи на конкретных примерах обосновывается эффективность применения теоремы при решении нестандартных геометрических задач на математических олимпиадах. Результаты исследования демонстрируют неразрывную связь геометрии и тригонометрии и способствуют формированию целостного представления о математических закономерностях.

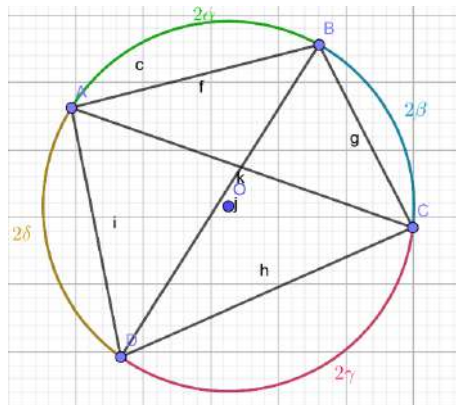
Ключевые слова: теорема Птолемея, соотношения хорд и дуг, теорема синусов, тригонометрическое тождество, циклические (вписанные) четырёхугольники, планиметрическое моделирование, задача и решение.

Geometriya tarixida Ptolemey teoremasi matematikaning turli sohalarini bog‘lovchi fundamental natijalardan biri hisoblanadi. Milodiy II asrda yashagan Klavdiy Ptolemey nomi bilan ataluvchi ushbu teorema, dastlab astronomik vatarlar jadvalini tuzish uchun xizmat qilgan bo‘lsa, bugungi kunda planimetriya va trigonometriyaning uzviy sinteziga aylangan.

Ptolemey teoremasi aylanaga ichki chizilgan to‘rtburchaklar tomonlari va diagonallari orasidagi universal bog‘liqlikni ifodalaydi. Ushbu maqolaning dolzarbligi shundaki, teorema nafaqat statik geometrik formula, balki murakkab trigonometrik ayniyatlarni keltirib chiqarish va olimpiada darajasidagi nostandart masalalarni yechish imkonini beruvchi dinamik instrumentdir. Xususan, burchaklar yig‘indisi uchun sinuslar formulasini vatar uzunliklari orqali isbotlash uning analitik quvvatini ko‘rsatadi.

Ushbu ishda teoremaning Sinuslar qonuniga asoslangan trigonometrik isboti tahlil qilinadi va uning vatar-yoy munosabatlari orqali fundamental ayniyatlarni shakllantirishdagi o'rni o'rganiladi. Shuningdek, teoremaning matematika olimpiadalaridagi amaliy samaradorligi konkret misollar bilan asoslanadi. Maqsad – Ptolemey teoremasining ko'p qirrali ekanligini ko'rsatish va geometriya hamda trigonometriyaning yaxlitligini dalillashdan iborat.

Ptolemey teoremasi: Aylanaga ichki chizilgan to'rtburchak diagonallarining ko'paytmasi uning qarama-qarshi tomonlari ko'paytmalarining yig'indisiga teng, ya'ni $AC \cdot BD = AB \cdot CD + BC \cdot AD$



1-rasm. Teoremaga mos shakl

Isbot. Aylanaga ichki chizilgan $ABCD$ to'rtburchakni olaylik. Aylananing radiusi R bo'lsin. To'rtburchakning tomonlari va diagonallarini yoy burchaklari orqali ifodalaymiz.

$$\widehat{AB}=2\alpha, \widehat{BC}=2\beta, \widehat{CD}=2\gamma, \widehat{AD}=2\delta$$

Aylana to'liq 360° bo'lgani uchun:

$$2\alpha+2\beta+2\gamma+2\delta=360^\circ$$

$$\alpha+\beta+\gamma+\delta=180^\circ$$

Ixtiyoriy vatar uzunligi $L = 2R \sin \frac{\theta}{2}$ formulasi bilan hisoblanadi (bu yerda θ — vatar tiralgan yoy).

Shunga ko'ra:

$$AB = 2R \sin \alpha, \quad BC = 2R \sin \beta, \quad CD = 2R \sin \gamma, \quad AD = 2R \sin \delta$$

Diagonallar uchun ham xuddi shunday:

$$AC \text{ diagonali } AB \text{ va } BC \text{ yoylariga tiralgan, demak: } AC = 2R \sin(\alpha+\beta)$$

$$BD \text{ diagonali } BC \text{ va } CD \text{ yoylariga tiralgan, demak: } BD = 2R \sin(\beta+\gamma)$$

Isbotlashimiz kerak bo'lgan formula:

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

Topilgan qiymatlarni o'rniga qo'yamiz ($4R^2$ ga qisqartirib yuboramiz):

$$\sin \alpha \sin \gamma + \sin \beta \sin \delta = \sin(\alpha+\beta) \sin(\beta+\gamma)$$

Biz bilamizki, $\delta = 180^\circ - (\alpha+\beta+\gamma)$. Demak, $\sin \delta = \sin(\alpha+\beta+\gamma)$ Chap tomondagi ifodani ko'paytmani ayirmaga o'tkazish orqali yozsak, natijada o'ng tomondagi ko'paytmaga teng ekanligi isbotlanadi.

$$\sin \alpha \sin \gamma = \frac{1}{2} [\cos(\alpha-\gamma) - \cos(\alpha+\gamma)]$$

$$\sin \beta \sin(\alpha+\beta+\gamma) = \frac{1}{2} [\cos(\alpha+\gamma) - \cos(\alpha+2\beta+\gamma)]$$

$$\sin \alpha \sin \gamma + \sin \beta \sin(\alpha+\beta+\gamma) = \frac{1}{2} [\cos(\alpha-\gamma) - \cos(\alpha+\gamma) + \cos(\alpha+\gamma) - \cos(\alpha+2\beta+\gamma)] =$$

$$\frac{1}{2} [\cos(\alpha-\gamma) - \cos(\alpha+2\beta+\gamma)]$$

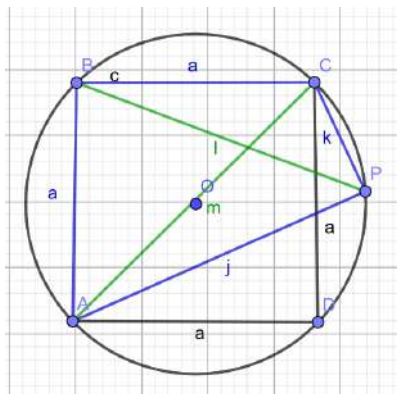
$$\sin(\alpha+\beta) \sin(\beta+\gamma) = \frac{1}{2} [\cos((\alpha+\beta)-(\beta+\gamma)) - \cos((\alpha+\beta)+(\beta+\gamma))] = \frac{1}{2} [\cos(\alpha-\gamma) - \cos(\alpha+2\beta+\gamma)]$$

teorema isbotlandi.

Ushbu isbot Ptolemey teoremasi va trigonometriya o'rtasidagi uzviy bog'liqlikni ko'rsatadi. Xususan, agar biz AC ni diametr deb olsak ($AC = 2R$), u holda $\alpha + \beta = 90^\circ$ bo'ladi va bu formula to'g'ridan-to'g'ri sinuslar yig'indisi formulasiga aylanadi: $\sin(\beta + \gamma) = \sin \beta \cos \gamma + \sin \gamma \cos \beta$

1-masala. $ABCD$ kvadratga tashqi chizilgan aylananing CD yoyida ixtiyoriy P nuqta olingan. Quyidagi tenglikni isbotlang: $PA + PC = \sqrt{2}PB$

Yechim: A, B, C, D nuqtalar kvadrat uchlari bo'lgani uchun ular bitta aylanada yotadi. P nuqta ham shu aylanada yotganligi sababli, biz $ABCP$ to'rtburchagini ko'rib chiqamiz. Ushbu to'rtburchak aylanaga ichki chizilgan (siklik) to'rtburchakdir. $ABCP$ to'rtburchak uchun Ptolemey teoremasini qo'llasak $PA \cdot BC + PC \cdot AB = PB \cdot AC$ bo'ladi.



2-rasm. Masala shartiga mos shakl

Kvadratning barcha tomonlari teng, ya'ni $AB = BC = CD = AD = a$ $AC = \sqrt{2}a$. Ushbu tengliklarni yuqoridagi formulaga qo'yamiz:

$$PA \cdot a + PC \cdot a = PB \cdot \sqrt{2}a$$

Tenglikning ikkala tomonini a ga bo'lib yuramiz (chunki $a \neq 0$):

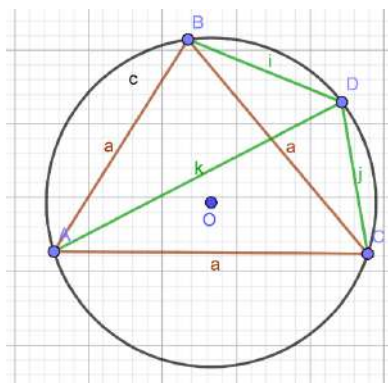
$$PA + PC = \sqrt{2}PB$$

Isbotlandi.

2-masala (Viloyat olimpiada masalasi 10-sinf 2008-yil). ABC muntazam uchburchakka tashqi chizilgan aylananing $\widehat{BC}$ yoyida (A nuqta yotmagan) ixtiyoriy D nuqta olingan. Quyidagi tenglikni isbotlang:

$$AD = BD + CD$$

Yechim. A, B, C va D nuqtalari bitta aylanada yotadi.



3-rasm. Masala shartiga mos shakl

Binobarin, $ABDC$ (yoki $ABCD$) to'rtburchak aylanaga ichki chizilgan to'rtburchakdir.

$$AB = AC = BC = a$$

$ABDC$ to'rtburchak uchun Ptolemey teoremasini qo'llaymiz:

$$AD \cdot BC = AC \cdot BD + AB \cdot CD$$

$$AD \cdot a = a \cdot BD + a \cdot CD$$

Tenglikning ikkala tomonini a ga bo'lib yuramiz (chunki $a \neq 0$):

$$AD = BD + CD$$

Tenglik isbotlandi.

3-masala: Agar $\alpha = \frac{\pi}{7}$ bo'lsa, tenglikni isbotlang: $\frac{1}{\sin \alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha}$

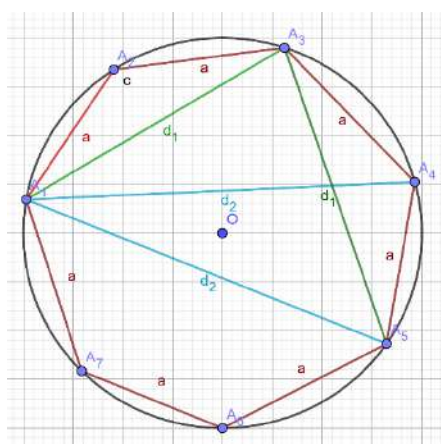
Yechish: Shu masalani yechish uchun muntazam yetti burchakli ko'pburchakni (yoki uning ma'lum bir qismini) aylanaga ichki chizilgan holda tasavvur qilamiz. Muntazam yetti burchakli $A_1A_2A_3A_4A_5A_6A_7$ berilgan bo'lsin. ularning barcha tomonlari a ga teng.

Diagonallarini belgilab olamiz: $d_1 = A_1A_3$, $d_2 = A_1A_4$

$A_1A_3A_4A_5$ to'rtburchak aylanaga ichki chizilgan (chunki u muntazam ko'pburchakning bir qismi).

Uning tomonlari: $A_1A_3 = d_1$, $A_3A_4 = a$,

$A_4A_5 = a$, $d_2 = A_1A_5$ diagonallari: $d_2 = A_1A_4$, $d_1 = A_3A_5$



4-rasm. Masala shartiga mos shakl

Ptolemey teoremasini qo'llaymiz:

$$A_1A_3 \cdot A_4A_5 + A_1A_5 \cdot A_3A_4 = A_1A_4 \cdot A_3A_5$$

$$d_1 \cdot a + d_2 \cdot a = d_2 \cdot d_1$$

Vatar uzunligini topish formulasidan $L = 2R \sin \frac{\theta}{2}$ (bu yerda θ – vatar tiralgan yoy) dan foydalanib quyidagi tengliklarga ega bo'lamiz:

$$a = 2R \sin \alpha, \quad d_1 = 2R \sin 2\alpha, \quad d_2 = 2R \sin 3\alpha$$

Ushbu tengliklarni ptolemey teoremasi natijasiga qo'yamiz va har bir hadni

$$(a \cdot d_1 \cdot d_2) \text{ ga bo'lamiz. Natijada } \frac{1}{\sin \alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha} \text{ tenglik hosil bo'ladi.}$$

Xulosa qilib aytganda, Ptolemey teoremasini umumta'lim hamda ixtisoslashtirilgan maktablar o'quv dasturiga integratsiya qilish matematik ta'lim sifatini oshirish yo'lida muhim qadamlardan biri bo'la oladi. Siklik (aylanaga ichki chizilgan) to'rtburchaklar mavzusini Ptolemey munosabati asosida o'rganish o'quvchilarda nostandart masalalarni yechish ko'nikmalarini, mantiqiy va analitik tafakkurni izchil rivojlantiradi.

Mazkur yondashuv, ayniqsa, fan olimpiadalariga qiziquvchi yoshlar uchun alohida ahamiyat kasb etadi, chunki u murakkab geometrik muammolarni chuqurroq tushunish va samarali hal etish imkonini beradi. Shu bilan birga, bunday tayyorgarlik nafaqat milliy standart imtihonlarda muvaffaqiyat qozonish, balki kelgusida murakkab muhandislik masalalari hamda ilmiy modellashirish jarayonlarida mustahkam nazariy poydevor yaratishga xizmat qiladi.

Teoremaning ichki uyg'unligi, isbotning nafisligi va mantiqiy mukammalligi esa yosh avlodda aniq fanlarga nisbatan estetik did, ilmiy qiziqish va yuqori motivatsiyani shakllantiruvchi muhim omil hisoblanadi.

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MATRITSANING XARAKTERISTIK KO'PHADI. GAMILTON – KELI TEOREMASINING MASALALARNI YECHISHGA TATBIQI

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Annotatsiya. Ushbu ilmiy maqolada chiziqli algebra sohasining muhim tushunchalaridan biri bo'lgan matritsaning xarakteristik ko'phadi va uning xossalari batafsil tahlil qilinadi. Xususan, xarakteristik ko'phadni hosil qilish usullari, uning ildizlari orqali matritsaning xos qiymatlarini aniqlash masalalari yoritilgan. Shuningdek, Gamilton – Keli teoremasining mazmuni, isboti va amaliy ahamiyati keng ko'lamda ko'rib chiqiladi. Mazkur teoremaning matritsalar ustida hisoblashlarni soddalashtirishdagi o'rni hamda yuqori darajali matritsa darajalarini kamaytirishda qo'llanilishi misollar yordamida asoslab berilgan.

Maqolada nazariy natijalarni amaliy masalalarni yechishga tatbiq etish masalalariga alohida e'tibor qaratilgan bo'lib, matritsa funksiyalarini hisoblash, matritsa darajalarini aniqlashda Gamilton – Keli teoremasidan foydalanish usullari tahlil qilingan. Olingan natijalar oliy ta'lim muassasalarida chiziqli algebra fanini o'qitishda, talabalarni turli olimpiadalarga tayyorlashda hamda ilmiy tadqiqotlarda samarali qo'llanishi mumkin.

Kalit so'zlar: matritsa, xarakteristik ko'phad, xos qiymatlar, determinant, Gamilton – Keli teoremasi, matritsa darajalari, chiziqli algebra, matritsa funksiyalari, algebraik ko'phad.

THE CHARACTERISTIC POLYNOMIAL OF A MATRIX. APPLICATION OF THE CAYLEY–HAMILTON THEOREM TO PROBLEM SOLVING

Abstract. This scientific article examines in detail one of the fundamental concepts of linear algebra—the characteristic polynomial of a matrix and its properties. In particular, methods for constructing the characteristic polynomial and determining the eigenvalues of a matrix through its roots are discussed. Furthermore, the content, proof, and practical significance of the Cayley–Hamilton theorem are comprehensively analyzed. The role of this theorem in simplifying matrix computations and reducing higher powers of matrices is substantiated with illustrative examples.

Special attention is given to the application of theoretical results to solving practical problems, including the computation of matrix functions and determining matrix powers using the Cayley–Hamilton theorem. The obtained results can be effectively applied in teaching linear algebra in higher education institutions, preparing students for various mathematical competitions, and conducting scientific research.

Keywords: matrix, characteristic polynomial, eigenvalues, determinant, Cayley–Hamilton theorem, matrix powers, linear algebra, matrix functions, algebraic polynomial.

ХАРАКТЕРИСТИЧЕСКИЙ МНОГОЧЛЕН МАТРИЦЫ. ПРИМЕНЕНИЕ ТЕОРЕМЫ ГАМИЛЬТОНА–КЭЛИ К РЕШЕНИЮ ЗАДАЧ

Аннотация. В данной научной статье подробно рассматривается одно из важных понятий линейной алгебры — характеристический многочлен матрицы и его свойства. В частности, освещаются методы построения характеристического многочлена, а также вопросы определения собственных значений матрицы через его корни. Кроме того, всесторонне анализируются содержание, доказательство и практическое значение теоремы Кэли–Гамильтона. Роль данной теоремы в упрощении вычислений над матрицами, а также в понижении степеней матриц высокого порядка обоснована на примерах.

Особое внимание уделено применению теоретических результатов к решению практических задач, в том числе вычислению матричных функций и определению степеней матриц с использованием теоремы Кэли–Гамильтона. Полученные результаты могут эффективно применяться в преподавании линейной алгебры в высших учебных заведениях, при подготовке студентов к различным олимпиадам, а также в научных исследованиях.

Ключевые слова: матрица, характеристический многочлен, собственные значения, определитель, теорема Кэли–Гамильтона, степени матриц, линейная алгебра, матричные функции, алгебраический многочлен.

Kirish. Zamonaviy matematikaning muhim yo'nalishlaridan biri bo'lgan chiziqli algebra nafaqat nazariy, balki amaliy jihatdan ham katta ahamiyat kasb etadi. Xususan, matritsalar nazariyasi ko'plab ilmiy va muhandislik masalalarini yechishda asosiy vositalardan biri sifatida xizmat qiladi. Matritsalar yordamida chiziqli tenglamalar sistemalarini yechish, geometrik o'zgartirishlarni ifodalash, iqtisodiy va texnik jarayonlarni modellashtirish kabi masalalar samarali hal etiladi.

Matritsa nazariyasining markaziy tushunchalaridan biri bu xarakteristik ko'phad hisoblanadi. Xarakteristik ko'phad orqali matritsaning muhim invariantlari – xos qiymatlari aniqlanadi. Ushbu qiymatlar esa o'z navbatida matritsaning strukturaviy xossalarini, uning diagonalizatsiya qilinish imkoniyatini va boshqa muhim algebraik jihatlarini ochib beradi. Shu sababli xarakteristik ko'phadni o'rganish chiziqli algebra kursining ajralmas qismi hisoblanadi.

Matritsalar nazariyasida alohida o'rin tutadigan muhim natijalardan biri Gamilton – Keli teoremasidir. Ushbu teorema har qanday kvadrat matritsa o'zining xarakteristik ko'phadini qanoatlantirishini ta'kidlaydi. Bu natija nafaqat nazariy jihatdan muhim, balki amaliy hisoblashlarda ham katta qulayliklar yaratadi. Xususan, yuqori darajali matritsa darajalarini kamaytirish, matritsa funksiyalarini hisoblash va murakkab algebraik ifodalarni soddalashtirishda ushbu teorema samarali qo'llaniladi.

Bugungi kunda Gamilton – Keli teoremasining qo'llanish sohasi kengayib bormoqda. U differensial tenglamalar nazariyasida, boshqaruv tizimlarida, fizik jarayonlarni modellashtirishda va kompyuter grafikasi kabi zamonaviy yo'nalishlarda ham muhim ahamiyatga ega. Shu bois mazkur teoremani chuqur o'rganish va uni amaliy masalalarga tatbiq etish dolzarb masalalardan biri hisoblanadi.

Ushbu maqolaning asosiy maqsadi matritsaning xarakteristik ko'phadi tushunchasini chuqur tahlil qilish, uning asosiy xossalarini yoritish hamda Gamilton – Keli teoremasining mazmuni va amaliy qo'llanishlarini ochib berishdan iboratdir. Shuningdek, teoremani konkret masalalarni yechishda qo'llash usullari misollar asosida ko'rsatib beriladi.

Adabiyotlar tahlili va metodlar. Chiziqli algebra, xususan, matritsalar nazariyasi va ularning xarakteristik ko'phadlari ko'plab mahalliy va xorijiy olimlar tomonidan chuqur o'rganilgan. Ushbu yo'nalishdagi fundamental natijalar klassik algebra maktabi vakillari tomonidan ishlab chiqilgan bo'lib, ular hozirgi zamon matematikasi rivojida muhim o'rin tutadi.

Xususan, xarakteristik ko'phad va xos qiymatlar tushunchalari dastlab matritsalar nazariyasining shakllanish davridayoq keng tadqiq qilingan. Bu borada chiziqli almashtirishlarning spektral xossalarini o'rganish asosiy yo'nalishlardan biri bo'lib, natijada matritsani diagonal ko'rinishga keltirish, uning kanonik shakllarini aniqlash masalalari ishlab chiqilgan. Klassik darsliklarda xarakteristik ko'phadni determinant orqali aniqlash usuli, uning ildizlari sifatida xos qiymatlarni topish jarayonlari batafsil yoritilgan.

Gamilton – Keli teoremasi esa matritsalar nazariyasining eng muhim natijalaridan biri sifatida ko'plab ilmiy manbalarda alohida o'rin egallaydi. Ushbu teorema XIX asrda mustaqil ravishda isbotlangan bo'lib, keyinchalik uning turli xil isbotlari va umumlashmalari taklif etilgan. Zamonaviy adabiyotlarda teoremaning determinantlar, adjungirlangan matritsa hamda minimal ko'phad tushunchalari orqali isbotlari keng yoritilgan. Bundan tashqari, uning sonli usullar va hisoblash matematikasidagi qo'llanilishiga ham alohida e'tibor qaratilgan.

So'nggi yillarda olib borilgan tadqiqotlarda Gamilton–Keli teoremasining amaliy jihatlari, xususan, yuqori darajali matritsa darajalarini hisoblashni soddalashtirish, matritsa eksponentasi va boshqa funksiyalarni aniqlashdagi roli keng tahlil qilinmoqda. Shuningdek, ushbu teoremaning differensial tenglamalar sistemalarini yechishdagi ahamiyati ham ko'plab ilmiy ishlarda ko'rsatib o'tilgan.

Mazkur maqolada yuqorida qayd etilgan ilmiy natijalarga tayangan holda, matritsaning xarakteristik ko'phadi va Gamilton – Keli teoremasining amaliy qo'llanilishiga doir masalalar tizimli ravishda o'rganildi.

Tadqiqotning metodologik asosi sifatida chiziqli algebra, matritsalar nazariyasi va algebraik ko'phadlar nazariyasining fundamental qoidalari xizmat qildi. Shuningdek, ilmiy izchillik, mantiqiylik va tizimlilik tamoyillariga qat'iy amal qilindi.

Natijalar va muhokama. Mazkur tadqiqot davomida matritsaning xarakteristik ko'phadi va Gamilton – Keli teoremasining nazariy asoslari tizimli ravishda o'rganildi hamda ularning amaliy masalalarni yechishdagi imkoniyatlari tahlil qilindi. Shuningdek, Gamilton – Keli teoremasining amaliy qo'llanilish imkoniyatlari keng yoritildi. Xususan, har qanday kvadrat matritsa o'zining xarakteristik ko'phadini qanoatlantirishi asosida yuqori darajali matritsa darajalarini past darajali ifodalar orqali yozish mumkinligi asoslandi. Bu natija hisoblash jarayonlarini sezilarli darajada soddalashtirish imkonini beradi. Tadqiqot

davomida ko'rsatildiki, ushbu yondashuv ayniqsa katta o'lchamli matritsalar bilan ishlashda samarali hisoblanadi.

Ushbu $P(x) = \det(xI_n - A)$ ko'phadga $A \in M_n(\mathbb{C})$ Matritsaning xarakteristik ko'phadi deyiladi. Ravshanki, xarakteristik ko'phadning ildizlari A matritsaning $\lambda_1, \lambda_2, \dots, \lambda_n$ xos sonlaridan iborat. Bu yerda x ni kompleks son shaklda ham yozsak bo'ladi. Ko'rinib turibdiki, $P(x)$ ko'phad ushbu shaklda $\mathbb{C}$ to'plamni $\mathbb{C}$ to'plamga akslantiradi. Agar bu yerdagi x o'rniga son emas matritsa qo'ysak ravshanki endi $P(x)$ ko'phad $M_n(\mathbb{C})$ to'plamni $M_n(\mathbb{C})$ ga akslantiradi va uning shakli ham $P(X) = (X - \lambda_1 I)(X - \lambda_2 I) \cdots (X - \lambda_n I)$ shaklda bo'ladi.

$A \in M_2(\mathbb{R})$ matritsa xarakteristik ko'phadining umumiy ko'rinishi:

$$|xI - A| = 0 \quad \text{yoki} \quad \lambda^2 - x_1\lambda + x_2 = 0.$$

Endi mavzuga oid bir nechta masalalarni yechimlari bilan keltirib o'tamiz.

1-masala. Berilgan matritsa xarakteristik ko'phadini tuzing: $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

Yechim. $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ matritsa xarakteristik ko'phadining umumiy ko'rinishi quyidagicha:

$\lambda^2 - x_1\lambda + x_2 = 0$. Bu yerda $x_1 = 1 + 2 = 3$ (bosh diagonal elementlari yig'indisi), $x_2 = \det A = 1 \cdot 2 - 0 \cdot 2 = 2$ (A matritsaning determinanti). Demak xarakteristik ko'phadning umumiy ko'rinishi quyidagicha ekan: $\lambda^2 - 3\lambda + 2 = 0$.

$A \in M_3(\mathbb{R})$ matritsa xarakteristik ko'phadining umumiy ko'rinishi:

$$|xI - A| = 0 \quad \text{yoki} \quad \lambda^3 - x_1\lambda^2 + x_2\lambda - x_3 = 0.$$

2-masala. Berilgan matritsaning xarakteristik ko'phadini tuzing:

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Yechim. $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ matritsaning xarakteristik ko'phadining umumiy ko'rinishi

quyidagicha: $\lambda^3 - x_1\lambda^2 + x_2\lambda - x_3 = 0$. Bu yerda $x_1 = 8 + 7 + 3 = 18$ (bosh diagonal elementlari yig'indisi),

$x_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} = 5 + 20 + 20 = 45$ (bosh diagonal elementlari kombinatsiyasidagi 2×2

matritsalar determinantlari yig'indisi), $x_3 = \det A = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = -60 + 40 + 20 = 0$ (A - matritsaning

determinanti). Demak xarakteristik ko'phadning umumiy ko'rinishi quyidagicha ekan: $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

3-masala. Berilgan matritsaning xarakteristik ko'phadini tuzing: $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ va $P(A) = O$ ni tekshiring.

Yechim. $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ matritsaning xarakteristik ko'phadini $|xI - A| = 0$ bu ko'rinishda qidirsak.

$$|xI - A| = \left| x \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right| = \begin{vmatrix} x-1 & -2 \\ -3 & x-4 \end{vmatrix} = (x-1) \cdot (x-4) - 6 = x^2 - 5x - 2 = 0.$$

Endi $P(A) = O$ mi tekshirsak:

$$P(A) = A^2 - 5A - 2I = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - 5 \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 10 & 22 \end{pmatrix} - \begin{pmatrix} 5 & 15 \\ 10 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Teorema. (Gamilton – Keli teoremasi) $P(x)$ A matritsaning xarakteristik ko‘phadi bo‘lsa $P(A) = O$ tenglik o‘rinli. Bu yerda O A ning o‘lchamiga mos nol matritsa.

Isbot. Ta’rifga ko‘ra $P(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I)$. Biz o‘ng tomondagi matritsalar ko‘paytmasini 0 ga tengligini ko‘rsatishimiz kerak. Schur uchburchak teoremasiga ko‘ra biz A ni $A = UTU^*$ shaklga keltira olamiz (U unitar, T yuqori uchburchakli matritsalar). Bundan foydalansak

$$\begin{aligned} P(A) &= (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = \\ &= (UTU^* - \lambda_1 I)(UTU^* - \lambda_2 I) \cdots (UTU^* - \lambda_n I) = \\ &= (UTU^* - \lambda_1 UU^*)(UTU^* - \lambda_2 UU^*) \cdots (UTU^* - \lambda_n UU^*) = \\ &= U(T - \lambda_1 I)U^*U(T - \lambda_2 I)U^*U(T - \lambda_3 I)U^* \cdots U(T - \lambda_n I)U^* = \\ &= U(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_n I)U^* = \end{aligned}$$

T ning yuqori uchburchakli matritsa ekanligini hisobga olsak, bilamizki T ning diagonalida $\lambda_1, \lambda_2, \dots, \lambda_n$ xos sonlar joylashadi.

$$\begin{aligned} &= U \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} U^* = \\ &= U \begin{pmatrix} 0 & 0 & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} U^* = \cdots = \\ &= U \begin{pmatrix} 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} U^* = U \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} U^* = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{aligned}$$

Teskari matritsasini topish. A - $n \times n$ o‘lchamli kvadrat matritsa va $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ uning xarakteristik ko‘phadi bo‘lsin. U holda $p(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = 0$, ya’ni matritsa o‘zining xarakteristik ko‘phadini qanoatlantiradi. Agar A matritsaning xarakteristik ko‘phadida $c_0 \neq 0$ bo‘lsa, $p(A) = 0$ dan $A(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1I) = -c_0I$ demak, $A^{-1} = \frac{1}{-c_0}(A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1I)$. Bu formula teskari matritsani topishning samarali usulidir.

4-masala. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ matritsaning teskarisini toping.

Yechim. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ matritsa uchun $p(\lambda) = (\lambda - 1)^2 - 1 = \lambda^2 - 4\lambda + 3$. U holda, $A^2 - 4A + 3I = 0$. Bundan esa $A^{-1} = \frac{1}{3}(4I - A) = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ ekanligi kelib chiqadi.

Matritsaning darajasini hisoblash. $P(A)=0$ tenglik A^n ni $I, A, \dots, A^{n-1}$ orqali ifodalash imkonini beradi. Shuning uchun istalgan $k \geq n$ uchun A^k ni shu asosda chiziqli ifodalash mumkin. Masalan, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ uchun $p(\lambda) = \lambda^2 - 1$, shuning uchun $A^2 = I, A^3 = A, A^4 = I$ va hakazo. Bu siklik xususiyat olimpiada masalalarida A^{2026} kabi ifodalarni soddalashtirishda yordam beradi.

Iz va determinantni topish. Xarakteristik ko'phadning koeffitsiyentlari iz va determinant bilan bog'liq: $c_{n-1} = -tr(A), c_0 = (-1)^n det(A)$. Gamilton-Keli teoremasi orqali bu kattaliklarni matritsali tenglamalardan topish mumkin. Masalan. Agar $A^2 = 2A - 3I$ bo'lsa, $p(\lambda) = \lambda^2 - 2\lambda + 3$ ekanligidan $tr(A) = 2, det(A) = 3$, kelib chiqadi.

5-masala. A va B 2×2 matritsalarining determinanti 1 ga teng bo'lsin. U holda quyidagini isbotlang:

$$tr(AB) - tr(A)tr(B) + tr(AB^{-1}) = 0.$$

Yechim. Gamilton – Keli teoremasiga ko'ra

$$B^2 - tr(B)B + I_2 = O_2.$$

Chap tomondan AB^{-1} ga ko'paytirsak:

$$AB - tr(B)A + AB^{-1} = O_2$$

Natijani olish uchun ikkala tomonni ham izini olib yuborsak.

$$tr(AB) - tr(A)tr(B) + tr(AB^{-1}) = 0.$$

6-masala. A va B 3×3 matritsalar bo'lsin. Isbotlang:

$$det(AB - BA) = \frac{tr((AB - BA)^3)}{3}.$$

Yechim. Gamilton – Keli teoremasiga ko'ra

$$(AB - BA)^3 - c_1(AB - BA)^2 + c_2(AB - BA) - c_3I_3 = O_3$$

Bu yerda $c_1 = tr(AB - BA) = 0$ va $c_3 = det(AB - BA)$. Izini olib, $AB - BA$ ni izi 0 ekanligidan foydalanib,

$$tr((AB - BA)^3) - 3det(AB - BA) = 0$$

ni olamiz va tenglik isbotlandi.

Muhokama jarayonida aniqlanishicha, xarakteristik ko'phad va Gamilton–Keli teoremasi o'zaro uzviy bog'liq bo'lib, ularni birgalikda qo'llash matritsalar ustida bajariladigan ko'plab amallarni optimallashtirish imkonini beradi. Ayniqsa, murakkab algebraik ifodalarni soddalashtirish, rekurrent formulalarni hosil qilish hamda chiziqli operatorlarning xossalari o'rganishda ushbu yondashuv samarali ekanligi kuzatildi.

Umuman olganda, olib borilgan tadqiqot natijalari matritsalar nazariyasining muhim tushunchalari o'rtasidagi bog'liqlikni yanada chuqurroq ochib berdi hamda ularning amaliy masalalarni yechishdagi samaradorligini tasdiqladi. Ushbu natijalar chiziqli algebra fanini o'qitishda, talabalarni ilmiy-tadqiqot ishlariga jalb etishda va murakkab matematik masalalarni yechishda qo'llanishi mumkin.

Xulosa. Mazkur maqolada matritsaning xarakteristik ko'phadi va Gamilton – Keli teoremasining nazariy hamda amaliy jihatlari kompleks tarzda o'rganildi. Tadqiqot natijasida xarakteristik ko'phadlar yordamida matritsaning xos qiymatlarini aniqlash, uning asosiy algebraik xossalari tahlil qilish imkoniyati keng yoritildi.

Gamilton – Keli teoremasining mazmuni va ahamiyati to'liq ochib berilib, har qanday kvadrat matritsa o'zining xarakteristik ko'phadini qanoatlantirishi asosida yuqori darajali matritsa darajalarini soddalashtirish mumkinligi asoslandi. Ushbu yondashuv matritsalar ustida hisoblashlarni optimallashtirishda muhim vosita ekanligi amaliy misollar orqali tasdiqlandi.

Tadqiqot davomida Gamilton – Keli teoremasidan foydalanib matritsa darajalarini hisoblash, matritsa funksiyalarini aniqlash va murakkab algebraik ifodalarni soddalashtirishning samarali usullari ishlab chiqildi. Bu esa mazkur teoremaning nafaqat nazariy, balki amaliy jihatdan ham katta ahamiyatga ega ekanligini ko'rsatadi.

Shuningdek, olib borilgan tahlillar shuni ko'rsatdiki, xarakteristik ko'phad va Gamilton – Keli teoremasini uyg'un holda qo'llash chiziqli algebra masalalarini yechishda yuqori samaradorlikka erishish imkonini beradi. Ushbu natijalar oliy ta'lim tizimida chiziqli algebra fanini o'qitishda, ilmiy-tadqiqot ishlarida hamda turli amaliy masalalarni yechishda qo'llanishi mumkin.

Kelgusida mazkur yo'nalishda katta o'lchamli matritsalar uchun sonli usullarni takomillashtirish, kompyuter texnologiyalari asosida samarali algoritmlar ishlab chiqish dolzarb masalalardan biri bo'lib qoladi.

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**О ЗАДАЧЕ СО СВОБОДНОЙ ГРАНИЦЕЙ ДЛЯ КВАЗИЛИНЕЙНОГО
ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ РЕАКЦИИ-ДИФФУЗИИ В МОДЕЛЯХ
ПОПУЛЯЦИОННОЙ ДИНАМИКИ**

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Аннотация. В данной работе исследуется задача со свободной границей для квазилинейного параболического уравнения типа реакции-диффузии, возникающего в моделях математической экологии и популяционной динамики. Получены априорные оценки решений в пространствах Гёльдера, что позволяет установить существование и единственность решения рассматриваемой задачи. Особое внимание уделяется анализу асимптотического поведения решения и явлению распространения или исчезновения популяции во времени. Показано, что в зависимости от параметров модели решение может описывать как устойчивое распространение вида, так и его вымирание при больших значениях времени. Полученные результаты могут быть применены при исследовании различных экологических и биологических процессов, описываемых реакционно-диффузионными моделями.

Ключевые слова: свободная граница, квазилинейное параболическое уравнение, реакционно-диффузионная модель, априорные оценки, существование и единственность решения, математическая экология, динамика популяций.

**ON A FREE BOUNDARY PROBLEM FOR A QUASILINEAR PARABOLIC REACTION-
DIFFUSION EQUATION IN POPULATION DYNAMICS MODELS**

Abstract. In this work, the problem with a free boundary for a quasi-linear parabolic equation of the reaction-diffusion type arising in models of mathematical ecology and population dynamics is investigated. *A priori* estimates of solutions in Hölder spaces have been obtained, which allows us to establish the existence and uniqueness of the solution of the problem under consideration. Special attention is paid to the analysis of the asymptotic behavior of the solution and the phenomenon of population distribution or disappearance over time. It has been shown that, depending on the model parameters, the solution can describe both the stable distribution of the species and its extinction at large time values. The obtained results can be applied in the study of various ecological and biological processes described by reaction-diffusion models.

Keywords: free boundary, quasi-linear parabolic equation, reaction-diffusion model, *a priori* estimates, existence and uniqueness of the solution, mathematical ecology, population dynamics.

**POPULYATSIYA DINAMIKASI MODELLARIDA REAKSION-DIFFUZIYA TIPIDAGI
KVAZICHIZIQLI PARABOLIK TENGLAMA UCHUN ERKIN CHEGARA MASALASI HAQIDA**

Annotatsiya. Ushbu ishda matematik ekologiya va populyatsion dinamika modellarida yuzaga keladigan reaksiya-diffuziya tipidagi kvazichiziqli parabolik tenglama uchun noma'lum chegarali masala o'rganilgan. Gyolder fazolarida yechimlarning aprior baholari olingan, bu esa qaralayotgan masala yechimining mavjudligi va yagonaligini aniqlash imkonini beradi. Yechimning asimptotik xatti-harakati va populyatsiyaning vaqt o'tishi bilan tarqalishi yoki yo'qolishi hodisasini tahlil qilishga alohida e'tibor qaratilmoqda. Model parametrlariga qarab, yechim turning barqaror tarqalishini ham, vaqtning katta qiymatlarida uning yo'q bo'lib ketishini ham tavsiflashi mumkinligi ko'rsatilgan. Olingan natijalar reaksiya-diffuziya modellari bilan tavsiflanadigan turli xil ekologik va biologik jarayonlarni o'rganishda qo'llanilishi mumkin.

Kalit so‘zlar: erkin chegara, kvazichiziqli parabolik tenglama, reaksiy-diffuzion model, aprior baholar, yechimning mavjudligi va yagonaligi, matematik ekologiya, populyatsiyalar dinamikasi.

Введение. Среди многих моделей, предложенных до сих пор, виды распространяются с фиксированной постоянной скоростью, независимо от локальной среды. Однако в неоднородной среде виды не могут полностью игнорировать окружающую среду, фактически, они часто могут ощущать и реагировать на локальные экологические сигналы, перемещаясь к благоприятным местообитаниям, и эти перемещения обычно зависят от сочетания местных биотических и абиотических факторов, таких как климат, пища, хищники и т. д. [9]. На основании этого Belgacem и Cosner [1] модифицировали, добавив член адвекции, и предложили следующую модель:

$$w_t - \nabla(d \nabla w - w \nabla m(x)) = w(m(x) - w) \text{ в } (0, \infty) \times \Omega,$$

где $\Omega \subseteq \mathbb{R}^N$ – ограниченная область; $m(x)$ представляет собой локальный темп роста популяции, который зависит от местоположения, $d \nabla w$ учитывает случайную диффузию, $w \nabla m(x)$ представляет собой движение вверх вдоль градиента окружающей среды.

После работы Belgacem и Cosner [1] появилось много теоретических разработок систем реакция-диффузия-адвекция. Качественные свойства неотрицательных решений данных систем при соответствующих граничных и начальных условиях были детально исследованы в работах [1, 2, 5-9].

Следуя [6-9], введем преобразование $u = e^{-\frac{m(x)}{d}} w$ и получим

$$u_t - d \Delta u - \nabla m \cdot \nabla u = u \left(m(x) - e^{\frac{m(x)}{d}} u \right).$$

Используя вышеуказанные работы, мы изучаем задачу со свободной границей для уравнения реакции-диффузии-адвекции в одномерном случае.

Постановка задачи. Требуется найти функции, $s(t)$, $u(t, x)$ в области $D = \{(t, x) : 0 < t \leq T, 0 < x < s(t)\}$, удовлетворяющие условиям

$$k(u)u_t - du_{xx} - m'(x)u_x = u \left(m(x) - e^{\frac{m}{d}} u \right), \quad (t, x) \in D, \quad (1)$$

$$u(0, x) = u_0(x), \quad 0 \leq x \leq s_0 \equiv s(0), \quad (2)$$

$$u_x(t, 0) = 0, \quad 0 \leq t \leq T, \quad (3)$$

$$u(t, s(t)) = 0, \quad 0 \leq t \leq T \quad (4)$$

$$s'(t) = -\mu u_x(t, s(t)), \quad 0 \leq t \leq T \quad (5)$$

где $m(x)$ представляет локальную скорость роста популяции, которая зависит от местоположения, $m'u_x$ – член адвекции, d, a, b, μ положительные константы. $k(u)$ – коэффициент биоёмкости среды. Начальная функция $u_0(x)$ удовлетворяет

$$u_0 \in C^2([0, s_0]), \quad u'_0 = u_0(s_0) = 0, u_0 > 0 \text{ в } [0, s_0), \quad \lim_{x \rightarrow s(t)} \frac{u_0(x)}{s_0 - x} = 0.$$

В этом параграфе мы предполагаем, что

$$\text{I. } m_1 \leq m(x) \leq m_2, \quad m(x) \in C^1([0, \infty)), \quad \tilde{M}_1 \leq m'(x) \leq \tilde{M}_2$$

$$\text{II. } k(u) \geq k_0 > 0 \text{ для любого } u > 0.$$

Сначала мы получаем априорные оценки типа Шаудера, которые **необходимые** для доказательства существования и единственности решения данной задачи. Наш подход основан на принципе максимума, принципе сравнения и построении подходящих верхних и нижних решений для (1)-(5).

Априорные оценки.

Теорема 1. Пусть функции $s(t)$, $u(t, x)$ являются решением задачи (1)-(5) и существует постоянная N такая, что $N \geq \max \left\{ \frac{u_0(x)}{s_0 - x}, \frac{M_1 m_2}{\tilde{M}_1} \right\}$. Тогда существуют положительные постоянные M_1 , M_2 , не зависящие от T , для которых справедливы оценки

$$0 < u(t, x) \leq M_1 \equiv \max \left(u_0, m_2 e^{\frac{m_1}{d}} \right), \quad (t, x) \in D, \quad (6)$$

$$0 < s'(t) \leq M_2, \quad 0 \leq t \leq T. \quad (7)$$

Доказательство. Доказательство похоже на доказательство теоремы 7.2 в [21].

Если в указанной теореме взять $\hat{\rho} = 0$, $\tilde{\rho} = \frac{a}{b}$, то для функции $f(t, x, u) = u \left(m(x) - e^{\frac{m}{d}} u \right)$ все условия теоремы выполняются. Тогда по утверждению указанной теоремы, имеем (6).

Далее, учитывая условие (5) и положительность функции $u(t, x)$ в области D , находим $u_x(t, s(t)) < 0$. Отсюда из (6) получаем $s'(t) > 0$.

Для получения верхней оценки $s'(t)$ построим вспомогательную функцию следующего вида:

$$v(t, x) = u(t, x) + N(x - s(t))$$

где N — положительная константа, зависящая от заданных функций.

Мы докажем, что $v_x(t, s(t)) \geq 0$ для соответствующего N . Простых вычислений получаем следующее

$$\begin{cases} k(v)v_t - dv_{xx} - m'(x)v_x \leq m_2 M_1 - \tilde{M}_1 N \leq 0, & (t, x) \in D, \\ v(0, x) = u_0(x) + N(x - s_0) \leq 0, & 0 \leq x \leq s_0, \\ v_x(t, 0) = N > 0, & 0 \leq t \leq T, \\ v(t, s(t)) = 0, & 0 \leq t \leq T. \end{cases}$$

Если $N \geq \max \left\{ \frac{u_0(x)}{s_0 - x}, \frac{M_1 m_2}{\tilde{M}_1} \right\}$, то по принципу максимума получаем $v(t, x) \leq 0$ в $\bar{D}$.

Тогда, учитывая $v(t, s(t)) = 0$ и лемму Хопфа, имеем $u_x(t, s(t)) \geq -N$. Последнее неравенство в сочетании с (5) дает $s'(t) \leq \mu N = M_2$ и доказательство завершено.

Теорема 1 доказана.

Начальные оценки получены. Теперь, используя результаты работы [14], устанавливаем априорные оценки для u_x и старших производных.

Сначала получим априорные оценки для $u(t, x)$. Из-за граничных условий (3)-(5) мы не можем использовать результаты работы [14]. Поэтому введем следующее преобразование

$$t = t, \quad y = \frac{x}{s(t)}$$

для выпрямления свободной границы. Тогда область D преобразуется в область $D_1 = \{(t, y) : 0 < t < T, 0 < y < 1\}$, и ограниченная функция $U(t, y) = u(t, x)$ является решением задачи

$$U_t - A(t, y, U)U_{yy} = B(t, y, U, U_y), \quad (t, y) \in D_1, \quad (8)$$

$$U(0, y) = U_0(y) \equiv u_0(s_0 y), \quad 0 \leq y \leq 1, \quad (9)$$

$$U_y(t, 0) = 0, \quad U(t, 1) = 0, \quad 0 \leq t \leq T, \quad (10)$$

где $D_1 = \{(t, y) : 0 < t \leq T, \quad 0 < y < 1\}, \quad A(t, y, U) = \frac{d}{s^2(t)}, \quad s'(t) = -\frac{\mu}{s(t)}U_y(t, 1),$

$$B(t, y, U, U_y) = \frac{f(U)}{k(U)} + \frac{k(U)ys'(t) + m_y}{k(U)s^2(t)}U_y, \quad f(U) = U \left(m - e^{\frac{m}{d}}U \right).$$

Теорема 2. Пусть $M_1 = \max_{\bar{D}_1} |u|$ и предположим, что $u(t, x), u_x(t, x)$ непрерывны в $\bar{D}_1$, а u удовлетворяет (1) в $\bar{D}_1$ вместе с (2)-(5). Тогда

$$|u_x(t, x)| \leq M_3(M_1, M_2), \quad (t, x) \in \bar{D}_1. \quad (11)$$

Более того, если слабые производные второго порядка (u_{xx}, u_{tx}) принадлежат $L^2(\bar{D}_1)$, то существует $\alpha = \alpha(M_1, \delta)$ такое, что

$$|u|_{1+\alpha}^{\bar{D}_1} \leq M_4(M_3). \quad (12)$$

Дополнительно предположим, что $u(t, x)$, удовлетворяющее (1) в $\bar{D}_1$, непрерывно вместе со своими производными u_t, u_x, u_{xx} и $|u|_{2+\alpha}^{\bar{D}_1} < \infty$.

Тогда

$$|u|_{2+\alpha}^{\bar{D}_1} \leq M_5(M_4). \quad (13)$$

Доказательство. Производная $(u_x(t, x))$ удовлетворяет однородному параболическому уравнению с ограниченными коэффициентами и ограничена в граничных точках $x = 0, x = s(t)$, а также при $x = 0$. Согласно принципу максимума, $u_x(t, x)$ ограничена в D_1 . Следовательно, (11) выполняется. Оценки (12) и (13) для $(t, x) \in D_1$ являются непосредственными следствиями результатов работы [14].

Функция $s(t)$ и правая часть (8)-(10) удовлетворяют условиям Гёльдера и теоремам. Применяя метод четного продолжения через правую границу [11], мы получаем (11), (12). Остальные оценки для производных более высокого порядка устанавливаются с использованием результатов для линейных уравнений [6, 9].

Дихотомия распространения-исчезновения. Этот раздел посвящен дихотомии распространения и исчезновения видов в окружающей среде.

Теорема 3. Пусть $(u(t, x), s(t))$ — решение задачи (1-5). Тогда справедливо следующее.

- **Исчезновение:** $\lim_{t \rightarrow \infty} s(t) = s_\infty \leq L$ и $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0$.
- **Распространение:** $s_\infty = \infty$ и $\lim_{t \rightarrow \infty} u(t, x) = U(x)$ равномерно по x на любом ограниченном

множестве из $[0, \infty)$, где $U(x)$ — единственное положительное решение следующей задачи

$$\begin{cases} -du_{xx} - m_x u_x = u \left(m(x) - e^{\frac{m}{d}} u \right), & 0 < x < \infty, \\ u_x(0) = 0. \end{cases}$$

Доказательство аналогично доказательству Теоремы 3.5 в [27].

Априорные оценки

$$|u_x^{\bar{D}}(t, x)| \leq M_3, \quad |u|_{1+\gamma}^{\bar{D}} \leq M_4, \quad |u|_{2+\gamma}^{\bar{D}} \leq M_5 \quad (14)$$

устанавливаются при помощи теорем 1 – 4 из [12]. Установленные априорные оценки позволяют доказать теоремы существования и единственности решения задачи (1)-(5).

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ON THE PERIODIC SOLUTION IN A NONLINEAR DYNAMICAL SYSTEM

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Abstract. This paper presents an analytical approximation of a closed trajectory within a two-dimensional nonlinear dynamical system. The existence of a stable limit cycle, a specific type of closed trajectory, for this system under particular parameter settings was rigorously demonstrated in prior research. This work focuses on the quantitative construction of this periodic solution. We represent the trajectory using a truncated Fourier series with $n = 8$ harmonics. The coefficients of this series are determined to give an explicit trigonometric form for the closed trajectory. To verify the accuracy of the constructed periodic solution, we compare the analytical solution with the exact limit cycle of the system.

Keywords: periodic solutions, nonlinear dynamical systems, Fourier series, numerical methods, harmonic balance.

CHIZIQSIZ DINAMIK SISTEMADAGI DAVRIY YECHIM HAQIDA

Annotatsiya. Ushbu maqolada ikki o'lchamli chiziqsiz dinamik sistemadagi yopiq trayektoriyaning analitik yaqinlashishi qurilgan. Avvalgi tadqiqotlarda sistemaning ma'lum parametrlari uchun $T \approx 10.65$ davrli limit davra mavjudligi aniqlangan. Ushbu qatorning koeffitsiyentlari yopiq trayektoriya uchun aniq trigonometrik shaklni berish maqsadida aniqlanadi. Biz taqribiy yechimni $n = 8$ garmonikali qisqartirilgan Furiye qatori yordamida ifodalaymiz. Taqribiy yechimning aniqligini tekshirish uchun analitik yechimni sistemaning limit davrasi bilan solishtiramiz.

Kalit so'zlar: davriy yechimlar, chiziqsiz dinamik sistema, Furiye qatorlari, sonli usullar, garmonik balans.

О ПЕРИОДИЧЕСКОМ РЕШЕНИИ В НЕЛИНЕЙНОЙ ДИНАМИЧЕСКОЙ СИСТЕМЕ

Аннотация. В данной работе строится аналитическое приближение замкнутой траектории в двумерной нелинейной динамической системе. Ранее для определенных параметров системы было установлено существование устойчивого предельного цикла с периодом $T \approx 10.65$. Мы представляем траекторию с использованием усеченного ряда Фурье с $n = 8$ гармониками. Коэффициенты этого ряда определяются таким образом, чтобы получить явную тригонометрическую форму для замкнутой траектории. Для проверки точности построенного периодического решения мы сравниваем аналитическое решение с точным предельным циклом системы.

Ключевые слова: периодические решения, нелинейной динамической систем, ряды Фурье, численные методы, гармонический баланс.

Introduction. Closed trajectories, also known as limit cycles, represent stable periodic behaviors in nonlinear dynamical systems. The study of these trajectories is central to understanding the dynamics of many systems in science and engineering.

Glycolytic oscillations have been extensively studied both experimentally and theoretically [1–4]. Mathematical modeling plays a central role in understanding the mechanisms responsible for such oscillatory dynamics. Although numerical simulations can accurately capture periodic behavior, they often do not provide explicit analytical insight.

Analytical methods, including the harmonic balance method, allow one to construct approximate representations of periodic solutions. In this work, we apply the harmonic balance method to a simplified glycolysis model to derive an approximate closed-form expression for its periodic orbit. This approach provides a clearer understanding of the structure of the oscillations and the influence of nonlinear interactions within the system.

Selkov introduced one of the first mathematical models, described by nonlinear equations, to represent this oscillatory behavior. In this model, x and y represent the concentrations of ADP and fructose-6-phosphate, respectively, while a and b are positive kinetic parameters. In dimensionless form, these equations are

$$\begin{aligned}\frac{dx}{dt} &= -x + ay + x^2y \\ \frac{dy}{dt} &= b - ay - x^2y\end{aligned}$$

The system demonstrates limit cycle oscillations, under certain conditions, a fact supported by the Poincaré-Bendixson theorem [5-6]. Although this theorem guarantees the existence of a limit cycle, it does not reveal its specific form or period.

For a specific set of parameter values, prior work [7] has rigorously proved the existence of a stable closed trajectory with a period T . This prior work confirmed the presence of sustained oscillations but did not provide an explicit analytical form for the trajectory.

Selkov model that exhibits a limit cycle with specific parameter values of the system above $a = 0.1$, $b = 0.5$ and passing near the point $(0.5400, 1.8186)$, with an approximate period of $T \approx 10.65$:

$$\dot{x} = -x + 0.1y + x^2y \quad (1.1)$$

$$\dot{y} = 0.5 - 0.1y - x^2y \quad (1.2)$$

This paper aims to construct such an explicit analytical approximation for the established closed trajectory. This constructive approach allows for a deeper quantitative understanding of the system's periodic behavior and demonstrates the effectiveness of harmonic balancing in providing precise analytical solutions for nonlinear dynamical systems.

Main result. We employ the Harmonic Balance Method (HBM), which represents the state variables as finite Fourier series. Substituting these series into the governing differential equations transforms the problem into solving a system of 17 nonlinear algebraic equations for the unknown Fourier coefficients.

We determine these coefficients by simultaneously solving this algebraic system for a truncation order of $N=8$ harmonics. The resulting trigonometric polynomial provides an analytical approximation of the closed trajectory.

In general, limit cycles can be approximately determined using a series of sine and cosine functions, as demonstrated in prior research [8-12].

$$x = b + a(t)\cos(\omega t) + \sum_{n=1}^m (C_n \cos(n\omega t) + D_n \sin(n\omega t)) \quad (2.1)$$

where b , C_n , and D_n are constants.

In the harmonic balance method, the goal is to find an approximate periodic solution to a nonlinear differential equation. The method involves assuming a solution in the form of a Fourier series and then substituting this series into the differential equation. This substitution results in a residual, which is a measure of how well the assumed solution satisfies the original equation.

The key idea behind setting the coefficients of constant terms (and other harmonics) in the residual to zero is to minimize this residual and thus obtain a good approximate solution [13].

Since the solution is periodic with period T , we can represent it as a Fourier series. In practice, we keep only a finite number of terms. We choose to keep terms up to the 8th harmonic. So we write:

$$x(t) = a_0 + \sum_{n=1}^8 [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \quad (2.2)$$

Here, $a_0, a_1, \dots, a_8, b_1, \dots, b_8$ are unknown constants that we need to find. There are 17 unknowns in total (one constant term a_0 , plus 8 cosine coefficients and 8 sine coefficients).

Notice that the first equation in our system can be rearranged to give y directly:

$$y = \frac{\dot{x} + x}{x^2 + 0.1} \quad (2.3)$$

If we have an expression for $x(t)$, we can compute its derivative and then find $y(t)$ using this formula. This is helpful because it means we only need to find the coefficients for x .

Our assumed solution (2.2) is only an approximation. When we use it in equation (2.3) to find y , and then substitute both into the second original equation, we will not get zero exactly. We define the residual $R(t)$ as:

$$R(t) = \dot{y} + 0.1y + x^2y - 0.5$$

If our approximation were exact, $R(t)$ would be zero for all t . Since it is not exact, $R(t)$ will be some small error function. The harmonic balance method gives us a way to make this error small in a systematic way.

The method requires that the residual $R(t)$ has no components at the frequencies we used in our approximation. Mathematically, this means that when we integrate $R(t)$ multiplied by each of our basis functions over one period, the result should be zero. This gives us the following equations:

$$\int_0^T R(t) dt = 0 \quad (2.4)$$

$$\int_0^T R(t) \cos(k\omega t) dt = 0, \quad k = 1, 2, \dots, 8 \quad (2.5)$$

$$\int_0^T R(t) \sin(k\omega t) dt = 0, \quad k = 1, 2, \dots, 8 \quad (2.6)$$

These are 17 equations (one from (2.4), eight from (2.5), and eight from (2.6)).

We also have the initial condition $x(0) = 0.5400$. Putting $t = 0$ into equation (2.2) gives:

$$x(0) = a_0 + a_1 + a_2 + \dots + a_8 = 0.5400 \quad (2.7)$$

This is one more equation.

Equations (2.4)-(2.7) together form a system of 18 equations for our 17 unknowns. This is an overdetermined system. However, the periodic solution is determined only up to a phase shift. To fix the phase uniquely, we replace one of these equations—specifically, the condition for the highest *sine* harmonic ($k = 8$)—with the initial condition.

In practice, we solve the system numerically using an iterative method. We look for values of $a_0, \dots, a_8, b_1, \dots, b_8$ that satisfy all equations as closely as possible. Modern numerical software can handle this type of problem effectively.

We solved the system of equations numerically. The solution gave us the values for all the coefficients. These are shown in Table 1.

a_n (Cosine)	b_n (Sine)
0.499967	-
0.115911	0.253287
-	0.047590
0.0579395	6
-	-
0.0196938	0.00902846
-	-
0.000567021	0.00675796
0.0017970	-
6	0.00128366
0.0006724	0.000279
32	589
0.0000375	0.000233
598	799

-	0.000045
0.0000621783	4077

Table 1: Fourier Coefficients for $x(t)$.

Notice that the coefficients become smaller as n increases. This is a good sign. It means that the lower frequency terms are the most important, and the higher frequency terms are just small corrections. This suggests that stopping at $n = 8$ gives a good approximation.

Using these coefficients, we can write the full approximate solution:

$$\begin{aligned} x(t) \approx & 0.499967 + 0.115911 \cos(\omega t) + 0.253287 \sin(\omega t) - 0.0579395 \cos(2\omega t) \\ & + 0.0475906 \sin(2\omega t) - 0.0196938 \cos(3\omega t) - 0.00902846 \sin(3\omega t) \\ & - 0.000567021 \cos(4\omega t) - 0.00675796 \sin(4\omega t) + 0.00179706 \cos(5\omega t) \\ & - 0.00128366 \sin(5\omega t) + 0.000672432 \cos(6\omega t) + 0.000279589 \sin(6\omega t) \\ & + 0.0000375598 \cos(7\omega t) + 0.000233799 \sin(7\omega t) - 0.0000621783 \cos(8\omega t) \\ & + 0.0000454077 \sin(8\omega t) \end{aligned}$$

where $\omega = 2\pi/10.65$.

These results tell us several things. First, the harmonic balance method has worked well for this problem. Second, the approximate solution we found is very close to the true periodic solution of the system. Third, this serves as independent confirmation that a periodic solution does indeed exist for these parameter values.

To verify the accuracy of the harmonic balance approximation, we compare the analytical solution (2.2) with the exact limit cycle of the system (1.1)–(1.2). The result is shown in Figure 1, where the exact trajectory is plotted in green and the approximate solution in red. The two curves coincide almost perfectly. This confirms that the truncated Fourier series with $n = 8$ provides an excellent approximation to the true periodic solution.

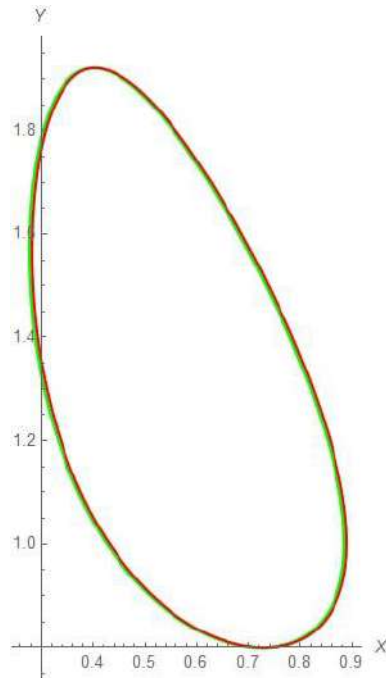


Figure 1. Comparison of the exact limit cycle (green) and the approximate periodic solution (red) obtained by the harmonic balance method

Conclusion. In this paper, we studied a non-linear dynamical system and looked for periodic solutions. Using the harmonic balance method, we constructed an approximate solution in the form of a Fourier series with 8 harmonics. We found the coefficients by solving a system of algebraic equations derived from the original differential equations and the initial condition.

The main results of this work are:

1. We obtained a complete set of Fourier coefficients (Table 1) that describe the periodic solution $x(t)$ for the given parameters.

2. Our results independently verify the existence of a periodic orbit for the parameter values $a = 0.1$ and $b = 0.5$, as previously suggested.

The harmonic balance method proved to be effective for this problem. It transformed a difficult differential equation problem into a more manageable algebraic problem. The main difficulty is that the algebraic equations are non-linear and must be solved numerically. However, modern computational tools make this feasible.

Future work could extend this analysis in several directions. One could include more harmonic terms to see if the accuracy improves further. One could also vary the parameters a and b to see how the periodic solution changes.

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UCHINCHI TARTIBLI TUZILMALI TENGLAMA UCHUN CHEGARAVIY MASALANI
REGULYARLASHTIRISH

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Annotatsiya. Ushbu ishda uchinchi tartibli tuzilmali tenglama uchun chegaraviy masala shartli korrektilikka o‘rganilgan. Masala yechimi yagonaligi va shartli turg‘unligi haqidagi teoremlar isbot qilingan. Korrektilik to‘plamida masalaning regulyarlashgan yechimi qurilgan.

Kalit so‘zlar: tuzilmali turdagi tenglama, nokorrekt masala, yagonalik, shartli turg‘unlik, regulyarlashgan yechim.

РЕГУЛЯРИЗАЦИЯ КРАЕВОЙ ЗАДАЧИ ДЛЯ СТРУКТУРНОГО УРАВНЕНИЯ
ТРЕТЬЕГО ПОРЯДКА

Аннотация. В данной работе исследуется условная корректность краевой задачи для структурного уравнения третьего порядка. Доказаны теоремы о единственности решения и его условной устойчивости. В множестве корректности построено регуляризованное решение задачи.

Ключевые слова: уравнение структурного типа, некорректная задача, единственность, условная устойчивость, регуляризованное решение.

REGULARIZATION OF A BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER
STRUCTURAL EQUATION

Abstract. This paper studies the conditional well-posedness of a boundary value problem for a third-order structural equation. Theorems on the uniqueness of the solution and its conditional stability are proved. A regularized solution of the problem is constructed within the set of correctness.

Keywords: structural-type equation, ill-posed problem, uniqueness, conditional stability, regularized solution.

Kirish. Zamonaviy xususiy hosilali tenglamalar nazariyasida tuzilmali turdagi tenglamalar, chiziqli va nochiziqli tuzilmali tenglamalar o‘ziga xos o‘rin tutadi. Bunday tenglamalar bilan bog‘liq chegaraviy masalalar ko‘pincha fizik jarayonlarning teskari masalalarini o‘rganishda qo‘llaniladi.

Ushbu ishda tuzilmali turdagi uchinchi tartibli differensial tenglama uchun chegaraviy masala shartli korrektilikka o‘rganilgan. Bu turdagi tenglamalar dissipativ jarayonlar yoki g‘ovakli muhitlarda to‘lqinni tavsiflash uchun ishlatiladi.

Masalaning qo‘yilishi: $\Omega = \{0 < x < l, 0 < t < T\}$ sohada

$$\left(\frac{\partial}{\partial t} + \alpha\right) \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) u = f(x, t) \quad (1)$$

tenglamani va

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad u_{tt}(x, 0) = h(x) \quad (2)$$

boshlang‘ich,

$$u(0, t) = u(l, t) = 0, \quad u_{xx}(0, t) = u_{xx}(l, t) = 0 \quad (3)$$

chegaraviy shartlarni qanoatlantiruvchi $u(x, t)$ funksiyani topish masalasini qaraymiz.

YECHIM QURISH

(1)-(3) masalani tadqiq qilish uchun

$$u_{tt} + u_{xx} = v(x, t)$$

belgilash kiritamiz. Natijada, $v(x, t)$ funksiyaga nisbatan

$$\begin{aligned} v_t + \alpha \cdot v &= f(x, t) \\ v(x, 0) &= h(x) + \varphi''(x) \end{aligned}$$

masalaga kelimiz. Bu yerdan

$$v(x, t) = \left(h(x) + \varphi''(x) + \int_0^t e^{\alpha \cdot t} \cdot f(x, \tau) d\tau \right) \cdot e^{-\alpha \cdot t} \quad (4)$$

bo'lishi aniqlanadi.

$$\text{Skalyar ko'paytma } (f(x), g(x)) = \int_0^l f(x) \cdot g(x) dx \text{ bo'lsin, u holda norma } \|f(x)\|^2 = \int_0^l f^2(x) dx$$

shaklida aniqlanadi.

(4) tenglikdan

$$\|v(x, t)\| \leq e^{-\alpha \cdot t} \cdot \left(\|h(x)\| + \|\varphi''(x)\| + \sqrt{\frac{e^{2\alpha \cdot t} - 1}{2 \cdot \alpha}} \cdot \left(\int_0^t \|f(x, \tau)\|^2 d\tau \right)^{1/2} \right) \quad (5)$$

tengsizlik kelib chiqadi.

$u(x, t)$ funksiyaga nisbatan esa quyidagi masalaga kelimiz:

$$u_{tt} + u_{xx} = v(x, t) \quad (6)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \quad (7)$$

$$u(0, t) = u(l, t) = 0 \quad (8)$$

(6)-(8) masala yechimini $u(x, t) = \bar{u}(x, t) + \vartheta(x, t)$ shaklda qidiramiz, bu yerda $\bar{u}(x, t)$:

$$\bar{u}_{tt} + \bar{u}_{xx} = 0$$

$$\bar{u}(x, 0) = \varphi(x), \quad \bar{u}_t(x, 0) = \psi(x)$$

$$\bar{u}(0, t) = \bar{u}(l, t) = 0$$

bir jinsli tenglamaga mos masalaning yechimi, $\vartheta(x, t)$ funksiya esa

$$\vartheta_{tt} + \vartheta_{xx} = v(x, t)$$

$$\vartheta(x, 0) = 0, \quad \vartheta_t(x, 0) = 0$$

$$\bar{u}(0, t) = \bar{u}(l, t) = 0$$

bir jinslimas tenglamaga mos masalaning yechimi bo'lsin.

Bir jinsli tenglamaga mos masalani yechish uchun Furiy usulidan foydalanamiz, ya'ni yechimni

$$\bar{u}(x, t) = X(t) \cdot T(t) \text{ ko'rinishda qidiramiz va } X_n(t) = \sin\left(\frac{\pi \cdot n}{l} \cdot x\right),$$

$$T_n(t) = A_n \operatorname{ch}\left(\frac{\pi \cdot n}{l} \cdot t\right) + B_n \operatorname{sh}\left(\frac{\pi \cdot n}{l} \cdot t\right), \quad \forall n \in N.$$

Bir jinsli tenglamaga mos masala yechimi quyidagi ko'rinishga ega bo'ladi:

$$\bar{u}(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cdot \operatorname{ch}\left(\frac{\pi \cdot n}{l} \cdot t\right) + \frac{l}{\pi \cdot n} \cdot \psi_n \cdot \operatorname{sh}\left(\frac{\pi \cdot n}{l} \cdot t\right) \right) \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

$$\text{bu yerda, } \varphi_n = \frac{2}{l} \cdot \int_0^l \varphi(x) \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right), \quad \psi_n = \frac{2}{l} \cdot \int_0^l \psi(x) \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right) dx$$

Endi bir jinslimas tenglamaga mos masalani yechamiz va yechimni

$$\vartheta(x, t) = \sum_{n=1}^{\infty} \frac{l}{\pi \cdot n} \cdot \int_0^t \operatorname{sh}\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot v_n(\tau) d\tau \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

shaklda ifodalash mumkin.

Demak, (1)-(3) masala yechimi ko‘rinishi quyidagicha bo‘ladi:

$$u(x, t) = \sum_{n=1}^{\infty} \left(\varphi_n \cdot \operatorname{ch} \left(\frac{\pi \cdot n}{l} \cdot t \right) + \frac{l}{\pi \cdot n} \cdot \psi_n \cdot \operatorname{sh} \left(\frac{\pi \cdot n}{l} \cdot t \right) + \frac{l}{\pi \cdot n} \cdot \int_0^t \operatorname{sh} \left(\frac{\pi \cdot n}{l} \cdot (t - \tau) \right) \cdot v_n(\tau) d\tau \right) \cdot \sin \left(\frac{\pi \cdot n}{l} \cdot x \right) \quad (9)$$

$$\text{bu yerda } v_n(t) = \frac{2}{l} \cdot \int_0^l v(x, t) \cdot \sin \left(\frac{\pi \cdot n}{l} \cdot x \right) dx$$

Yagonalik va shartli turg‘unlik teoremlari

1-lemma. (1) tenglamaning $u(0, t) = u(l, t) = 0$ shartlarni qanoatlantiruvchi yechimi uchun

$$\int_0^t \int_0^l u^2 dx d\tau \leq d(t) \left(T \int_0^l u^2(x, 0) dx + \gamma \right)^{1-r(t)} \left(\int_0^T \int_0^l u^2 dx dt + \gamma \right)^{r(t)} \quad (10)$$

$$\text{tengsizlik o‘rinli, } t \in (0, T), \quad \text{bu yerda } d(t) = e^{\frac{2T+1(1-e^{-2t})T-(1-e^{-2T})t}{2(1-e^{-2T})}},$$

$$\gamma = (2T^2 + 1) \int_0^T \int_0^l v^2(x, t) dx dt + |\alpha|, \quad \alpha = 2 \int_0^l (Tu_x^2(x, 0) - Tu_t^2(x, 0) + u(x, 0)u_t(x, 0)) dx,$$

$$r(t) = \frac{1 - e^{-2t}}{1 - e^{-2T}}.$$

Lemmaning isboti [8] adabiyotda keltirilgan.

(5) tengsizlikni hisobga olib, (10) tengsizlikka asosan

$$\int_0^t \int_0^l u^2 dx d\tau \leq d(t) \left(T \int_0^l u^2(x, 0) dx + \gamma_1 \right)^{1-r(t)} \left(\int_0^T \int_0^l u^2 dx dt + \gamma_1 \right)^{r(t)} \quad (11)$$

tengsizlik o‘rinli bo‘ladi, bu yerda

$$\gamma_1 = 3 \cdot (2T^2 + 1) \int_0^T e^{-2\alpha \cdot t} \cdot \left(\|h(x)\|^2 + \|\varphi''(x)\|^2 + \frac{e^{2\alpha \cdot t} - 1}{2 \cdot \alpha} \cdot \int_0^t \|f(x, \tau)\|^2 d\tau \right) dt + |\alpha|$$

(1)-(3) masalaning korrektlik to‘plami sifatida

$$M = \{u(x, t) : \|u(x, T)\| + \|u_{xx}(x, T)\| + \|u_t(x, T)\| \leq m, m < \infty\}$$

to‘plamni kiritamiz.

1-teorema. Faraz qilamiz, (1)-(3) masala yechimi mavjud va $u(x, t) \in M$ bo‘lsin. u holda (1)-(3) masala yechimi yagona.

Isbot. Aytaylik, $u_1(x, t)$ va $u_2(x, t)$ funksiyalar (1)-(3) masala yechimi bo‘lsin. $u(x, t) = u_1(x, t) - u_2(x, t)$ belgilash kiritamiz. U holda $u(x, t)$ funksiya

$$\left(\frac{\partial}{\partial t} + \alpha \right) (u_{tt} + u_{xx}) = f(x, t)$$

tenglamani,

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad u_{tt}(x, 0) = 0$$

boshlang‘ich,

$$u(0, t) = u(l, t) = 0, \quad u_{xx}(0, t) = u_{xx}(l, t) = 0$$

chegaraviy shartlarni qanoatlantiradi.

Bundan ko‘rinadiki, $\gamma_1 = 0$ va (11) tengsizlikka asosan $u(x, t) \equiv 0$ kelib chiqadi. Demak $u_1(x, t) \equiv u_2(x, t)$, ya‘ni (1)-(3) masala yechimi yagona ekan.

2-teorema. Faraz qilamiz, (1)-(3) masala yechimi mavjud va $u, u_\varepsilon \in M$ hamda $\|\varphi(x) - \varphi_\varepsilon(x)\|_{W_2^2} \leq \varepsilon$, $\|h(x) - h_\varepsilon(x)\| \leq \varepsilon$, $\|\psi(x) - \psi_\varepsilon(x)\| \leq \varepsilon$ $\max_t \|f(x, t) - f_\varepsilon(x, t)\| \leq \varepsilon$ bo'lsin.

U holda, $\int_0^t \|u - u_\varepsilon\|^2 d\tau \leq d(t) (T\varepsilon^2 + \gamma_\varepsilon)^{1-r(t)} (4Tm^2 + \gamma_\varepsilon)^{r(t)}$

tengsizlik o'rinli, bu yerda, $u_\varepsilon(x, t)$ - (1)-(3) masalaning $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$, $h_\varepsilon(x)$, $f_\varepsilon(x, t)$ boshlang'ich berilganlarga mos yechimi bo'lsin va

$$\gamma_\varepsilon = 2 \cdot (2T^2 + 1) \left(\frac{1}{8 \cdot \alpha^3} \cdot (1 - e^{2 \cdot \alpha \cdot T}) + \frac{T}{4 \cdot \alpha^2} (e^{2 \cdot \alpha \cdot T} - T \cdot \alpha) + 4 \cdot T \right) \varepsilon^2 + |\alpha|.$$

Isbot. $U(x, t) = u(x, t) - u_\varepsilon(x, t)$ belgilash kiritamiz. Unda $U(x, t)$ funksiya

$$\left(\frac{\partial}{\partial t} + \alpha \right) (U_{tt} + U_{xx}) = f(x, t) - f_\varepsilon(x, t)$$

tenglamani va

$$U(x, 0) = \varphi(x) - \varphi_\varepsilon(x), U_t(x, 0) = \psi(x) - \psi_\varepsilon(x), U_{tt}(x, 0) = h(x) - h_\varepsilon(x),$$

$$U(0, t) = U(l, t) = 0$$

shartlari qanoatlantiradi.

Ushbu masala uchun $U_{tt} + U_{xx} = V$ bo'lsin. U holda,

$$V_t + \alpha \cdot V = f(x, t) - f_\varepsilon(x, t),$$

$$V(x, 0) = h(x) - h_\varepsilon(x) + \varphi''(x) - \varphi_\varepsilon''(x),$$

hamda

$$U_{tt} + U_{xx} = V$$

$$U(x, 0) = \varphi(x) - \varphi_\varepsilon(x),$$

$$U_t(x, 0) = \psi(x) - \psi_\varepsilon(x),$$

$$U(0, t) = U(l, t) = 0$$

masalalarga kelimiz. (5) tengsizlikka ko'ra,

$$\|V(x, t)\| \leq \left(\|h(x) - h_\varepsilon(x)\| + \|\varphi''(x) - \varphi_\varepsilon''(x)\| + \sqrt{\frac{e^{2 \cdot \alpha \cdot t} - 1}{2 \cdot \alpha}} \cdot \left(\int_0^t \|f(x, \tau) - f_\varepsilon(x, \tau)\|^2 d\tau \right)^{1/2} \right) \cdot e^{-\alpha \cdot t} \leq \left(2 + \sqrt{\frac{(e^{2 \cdot \alpha \cdot t} - 1) \cdot t}{2 \cdot \alpha}} \right) \cdot \varepsilon.$$

Endi, (11) bahoga asosan

$$\alpha = 2T \int_0^l \left(\varphi'(x) - \varphi_\varepsilon'(x) \right)^2 dx - 2T \int_0^l \left(\psi(x) - \psi_\varepsilon(x) \right)^2 dx + 2 \int_0^l \left(\varphi(x) - \varphi_\varepsilon(x) \right) \left(\psi(x) - \psi_\varepsilon(x) \right) dx$$

bundan

$$|\alpha| \leq 2(2T + 1) \varepsilon^2.$$

$$\gamma = (2T^2 + 1) \int_0^T \|V(x, t)\|^2 dx + |\alpha| \leq \gamma_\varepsilon, \text{ bu yerda}$$

$$\gamma_\varepsilon = 2 \cdot (2T^2 + 1) \left(\frac{1}{8 \cdot \alpha^3} \cdot (1 - e^{2 \cdot \alpha \cdot T}) + \frac{T}{4 \cdot \alpha^2} (e^{2 \cdot \alpha \cdot T} - T \cdot \alpha) + 4 \cdot T \right) \varepsilon^2 + |\alpha|.$$

Yana

$$\int_0^T \|U(x, t)\|^2 dt = \int_0^T \|u(x, t) - u_\varepsilon(x, t)\|^2 dt \leq 4m^2.$$

Demak, $U = u - u_\varepsilon$ ekanligini inobatga olib, (11) tengsizlikdan

$$\int_0^t \|u - u_\varepsilon\|^2 d\tau \leq d(t) (T\varepsilon^2 + \gamma_\varepsilon)^{1-r(t)} (4Tm^2 + \gamma_\varepsilon)^{r(t)}.$$

talab qilingan tengsizlik kelib chiqadi.

Regulyarlashgan yechim qurish

1-hol. Faraz qilamiz $\varphi(x) = 0$, $\psi(x) = 0$, $h(x) = 0$, $f(x, t) \neq 0$, $u(x, t) \in M$ bo'lsin. U holda

$$v(x, t) = e^{-\alpha \cdot t} \cdot \int_0^t e^{\alpha \cdot \tau} \cdot f(x, \tau) d\tau \text{ va}$$

$$u(x, t) = \sum_{n=1}^{\infty} \frac{l}{\pi \cdot n} \cdot \int_0^t sh\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot e^{-\alpha \cdot \tau} \cdot \int_0^{\tau} e^{\alpha \cdot \tau_1} \cdot f(x, \tau_1) d\tau_1 d\tau \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

(1)-(3) masalaning regulyarlashgan yechimi sifatida

$$u_N(x, t) = \sum_{n=1}^N \frac{l}{\pi \cdot n} \cdot \int_0^t sh\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot e^{-\alpha \cdot \tau} \cdot \int_0^{\tau} e^{\alpha \cdot \tau_1} \cdot f_n(x, \tau_1) d\tau_1 d\tau \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

funksiyalar ketma-ketligini olamiz, bu yerda N - regulyarlashtirish parametri. Faraz qilamiz $\|f(x, t) - f_\varepsilon(x, t)\| \leq \varepsilon$ va $u(x, t) \in M$ bo'lsin. $f_\varepsilon(x, t)$ taqribiy berilganga mos regulyarlashgan yechim sifatida

$$u_N^\varepsilon(x, t) = \sum_{n=1}^N \frac{l}{\pi \cdot n} \cdot \int_0^t sh\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot e^{-\alpha \cdot \tau} \cdot \int_0^{\tau} e^{\alpha \cdot \tau_1} \cdot f_{\varepsilon n}(x, \tau_1) d\tau_1 d\tau \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

funksiyalar ketma-ketligini olamiz.

Ma'lumki,

$$\|u - u_N^\varepsilon\| \leq \|u - u_N\| + \|u_N - u_N^\varepsilon\|.$$

Oxirgi tengsizlikning o'ng tarafidagi ikkinchi qo'shiluvchini baholaymiz:

$$\|u_N - u_N^\varepsilon\| \leq \frac{\pi}{2} \cdot \sum_{n=1}^N \left(\frac{l}{\pi \cdot n}\right)^2 \cdot \left(\int_0^t sh\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot \int_0^{\tau} e^{\alpha \cdot (\tau_1 - \tau)} \cdot (f_n(\tau_1) - f_{\varepsilon n}(\tau_1)) d\tau_1 d\tau\right)^2$$

$$\text{bu yerda, } f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin\left(\frac{\pi \cdot n}{l} \cdot x\right) dx, \quad f_{\varepsilon n}(t) = \frac{2}{l} \int_0^l f_\varepsilon(x, t) \sin\left(\frac{\pi \cdot n}{l} \cdot x\right) dx$$

Bundan quyidagi tengsizlik kelib chiqishini ko'ramiz.

$$\|u_N - u_N^\varepsilon\| \leq \frac{l \cdot t \cdot \varepsilon}{8 \cdot \pi \cdot \alpha} \cdot \sqrt{\left(l \cdot sh\left(\frac{2 \cdot \pi \cdot N}{l} \cdot t\right) - 2 \cdot \pi \cdot t\right) \cdot (e^{-2 \cdot \alpha \cdot t} - 1 + 2 \cdot \alpha \cdot t)}$$

Endi $\|u - u_N\|$ ifodani baholaymiz:

$$\|u - u_N\| \leq \frac{\pi}{2} \cdot \sum_{n=N+1}^{\infty} \left(\frac{l}{\pi \cdot n}\right)^2 \cdot \left(\int_0^t sh\left(\frac{\pi \cdot n}{l} \cdot (t - \tau)\right) \cdot \int_0^{\tau} e^{\alpha \cdot (\tau_1 - \tau)} \cdot f_n(\tau_1) d\tau_1 d\tau\right)^2 \quad (12)$$

Ma'lumki $\|u(x, T)\| \leq m$ shartdan

$$\|u(x, T)\|^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{l}{\pi \cdot n}\right)^2 \cdot \left(\int_0^T sh\left(\frac{\pi \cdot n}{l} \cdot (T - \tau)\right) \cdot \int_0^{\tau} e^{\alpha \cdot (\tau_1 - \tau)} \cdot f_n(\tau_1) d\tau_1 d\tau\right)^2 \leq m^2 \quad (13)$$

kelib chiqadi. (13) shart ostida (12) qatorni baholash uchun shartli ekstremum Lagraj usulini qo'llaymiz va

$$g_n = \begin{cases} \sqrt{\frac{2}{\pi}} \cdot \frac{2 \cdot \pi \cdot n \cdot m}{l} \cdot e^{-\frac{\pi \cdot n}{l} T} + \bar{g}_n, n = N + 1 \\ 0, n \neq N + 1 \end{cases}$$

bo'lishini aniqlaymiz. bu yerda, $\bar{g}_n = \int_0^T \int_0^\tau e^{\alpha(\tau_1-\tau) + \frac{\pi \cdot n}{l}(\tau-2T)} \cdot f_n(\tau_1) d\tau_1 d\tau$. Natijada (12) qator uchun

$$\|u - u_N\| \leq \frac{l}{N+1} \cdot \sqrt{\frac{3}{8 \cdot \pi} \cdot \left(\frac{16 \cdot \pi \cdot (N+1)^2 \cdot m^2}{l^2} + 2 \cdot e^{\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot T} \cdot \bar{g}^2 + \frac{l \cdot (T^2 - t^2)}{8 \cdot \alpha \cdot N^2} \cdot p + \frac{t^2 \cdot l}{8 \cdot N^2} \cdot q \right)}$$

baho o'rinli, bu yerda

$$p = \frac{e^{-2 \cdot \alpha \cdot T + \frac{2 \cdot \pi \cdot (N+1)}{l} \cdot (t-T)} - e^{-2 \cdot \alpha \cdot T}}{\alpha \cdot l + \pi \cdot (N+1)} + \frac{1 - e^{\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot (t-T)}}{\pi \cdot (N+1)}, \quad q = \frac{1 - e^{-\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot t}}{\pi \cdot (N+1)} + \frac{e^{-\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot t} - e^{-2 \cdot \alpha \cdot t}}{\pi \cdot (N+1) - \alpha \cdot l}.$$

Demak,

$$\|u - u_N^\varepsilon\| \leq \frac{l \cdot t \cdot \varepsilon}{8 \cdot \pi \cdot \alpha} \cdot \sqrt{\left(l \cdot \operatorname{sh}\left(\frac{2 \cdot \pi \cdot N}{l} \cdot t\right) - 2 \cdot \pi \cdot t \right) \cdot (e^{-2 \cdot \alpha \cdot t} - 1 + 2 \cdot \alpha \cdot t) + \frac{l}{N+1} \cdot \sqrt{\frac{3}{8 \cdot \pi} \cdot \left(\frac{16 \cdot \pi \cdot (N+1)^2 \cdot m^2}{l^2} + 2 \cdot e^{\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot T} \cdot \bar{g}^2 + \frac{l \cdot (T^2 - t^2)}{8 \cdot \alpha \cdot N^2} \cdot p + \frac{t^2 \cdot l}{8 \cdot N^2} \cdot q \right)}}$$

tengsizlik o'rinli bo'ladi. Bundan

$$\inf_N \left\{ \frac{l \cdot t \cdot \varepsilon}{8 \cdot \pi \cdot \alpha} \cdot \sqrt{\left(l \cdot \operatorname{sh}\left(\frac{2 \cdot \pi \cdot N}{l} \cdot t\right) - 2 \cdot \pi \cdot t \right) \cdot (e^{-2 \cdot \alpha \cdot t} - 1 + 2 \cdot \alpha \cdot t) + \frac{l}{N+1} \cdot \sqrt{\frac{3}{8 \cdot \pi} \cdot \left(\frac{16 \cdot \pi \cdot (N+1)^2 \cdot m^2}{l^2} + 2 \cdot e^{\frac{2 \cdot \pi \cdot (N+1)}{l} \cdot T} \cdot \bar{g}^2 + \frac{l \cdot (T^2 - t^2)}{8 \cdot \alpha \cdot N^2} \cdot p + \frac{t^2 \cdot l}{8 \cdot N^2} \cdot q \right)}} \right\}$$

orqali regulyarlashtirish parametrik N aniqlanadi.

2-hol. Faraz qilamiz $\varphi(x) = 0$, $\psi(x) \neq 0$, $h(x) = 0$, $f(x, t) = 0$, $u(x, t) \in M$ bo'lsin. U holda

$$v(x, t) = (h(x) + \varphi''(x)) \cdot e^{-\alpha \cdot t},$$

$$u(x, t) = \sum_{n=1}^{\infty} \frac{l}{\pi \cdot n} \cdot \psi_n \cdot \operatorname{sh}\left(\frac{\pi \cdot n}{l} \cdot t\right) \cdot \sin\left(\frac{\pi \cdot n}{l} \cdot x\right)$$

shaklda ifodalanadi. Bu holda ham 1-holga o'xshab hisoblar bajarilganda

$$\|u - u_N^\varepsilon\| \leq \frac{m \cdot \operatorname{sh}\left(\frac{\pi \cdot (N+1)}{l} \cdot t\right)}{\operatorname{sh}\left(\frac{\pi \cdot (N+1)}{l} \cdot T\right)} + \frac{\varepsilon \cdot l}{\pi} \cdot \operatorname{sh}\left(\frac{\pi \cdot N}{l} \cdot t\right)$$

natija kelib chiqadi.

Xulosa. Ushbu ish tuzilmali turdagi differensial tenglama uchun chegaraviy masalaning shartli korrektiligini o'rganishga bag'ishlangan. Olib borilgan tadqiqotlardan shuni ko'rish mumkinki, ushbu masala Adamar ma'nosida nokorrekt hisoblanadi, olingan aprior baho asosida kiritilgan korrektilik to'plamida esa Tixonov ma'nosida shartli korrekt bo'ladi. Masalaning yechimi korrektilik to'plamida yagona va turg'unligini ifodalovchi teoremlar isbot qilindi. Shu tufayli, regulyarlashtirish parametrini aniq yechim bilan regulyarlashgan taqribiy yechim orasidagi farq bahosidan tanlash masalaning regulyarlashgan turg'un yechimini aniqlash imkonini beradi.

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CONIC SECTIONS: THEORY AND FORMULAE

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Abstract. This article provides information on conic sections and investigates their various forms. Conic sections have interested mathematicians since the era of Ancient Greece and remain a significant topic today. Over time, they began to be applied in fields such as astronomy and gained importance in technologies like radio telescopes, satellite dishes, and architecture. In the article, specialized software has been developed to visualize these curves, and the results are presented in the form of program output and graphical charts.

Keywords: Conic sections, analytic geometry, ellipse, parabola, hyperbola, circle, conic equations, focus, directrix, eccentricity.

KONIK KESIMLAR: NAZARIYA VA FORMULALAR

Annotatsiya. Ushbu maqolada konik kesimlar haqida ma'lumot berilgan va turli ko'rinishlari tadqiq etilgan. Konik kesimlar qadimgi Yunoniston davridan beri matematiklarni qiziqtirib kelmoqda va bugungi kunda ham muhim mavzu bo'lib qolmoqda. Vaqt o'tishi bilan konik kesimlar astronomiya kabi sohalarida qo'llanila boshlandi va radioteleskoplar, sun'iy yo'ldosh antennalari hamda arxitektura kabi texnologiyalarda muhim ahamiyat kasb etdi. Maqolada mazkur egri chiziqlarni vizuallashtirish uchun maxsus dasturiy ta'minot ishlab chiqilgan bo'lib, olingan natijalar dastur va grafik ko'rinishida keltirilgan.

Kalit so'zlar: Konik kesimlar, analitik geometriya, ellips, parabola, giperbola, aylana, konik tenglamalari, fokus, direktrisa, ekscentrisitet.

КОНИЧЕСКИЕ СЕЧЕНИЯ: ТЕОРИЯ И ФОРМУЛЫ

Аннотация. В данной статье представлена информация о конических сечениях и исследованы их различные виды. Конические сечения интересовали математиков со времен Древней Греции и по сей день остаются важной темой. Со временем они стали применяться в таких областях, как астрономия, и приобрели значимое место в таких технологиях, как радиотелескопы, спутниковые антенны и архитектура. В статье разработано специальное программное обеспечение для визуализации этих кривых, а полученные результаты представлены в виде программного кода и графиков.

Ключевые слова: Конические сечения, аналитическая геометрия, эллипс, парабола, гипербола, окружность, уравнения коник, фокус, директриса, эксцентриситет.

Introduction. Conic sections have interested mathematicians since ancient Greece and are still an important topic today. As early as 320 BCE, scholars like Menaechmus, Apollonius, and Archimedes studied these curves closely. Apollonius even wrote an eight-volume series about conic sections, creating ways to identify and study them with geometry. Over time, conic sections have been used in areas like astronomy and are important in technologies such as radio telescopes, satellite dishes, and architecture. The name “conic section” comes from how these curves are made: by cutting a cone with a plane. A cone has two identical parts called nappes, but usually, “cone” means just one nappe, which looks like a party hat. You can make a right circular cone by rotating a straight line through the origin around the y-axis.

Conic sections are curves obtained by intersecting a plane with two-napped cone. Depending on the angle of intersection, we get conic sections below:

- Circle
- Ellipse
- Parabola
- Hyperbola

These curves have important applications in architecture, astronomy, engineering, and many other important fields

Main Body.

Geometric Formation of Conic Sections.

Conic sections are formed by intersecting a plane with a double-napped cone. The type of section depends on the angle between the plane and the cone's axis (see Fig. 1).

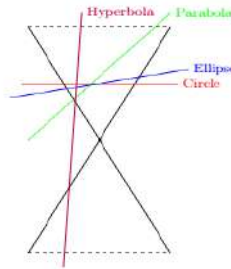


Figure 1: 2D representation of conic sections.

- Circle: Plane is perpendicular to the cone's axis.
- Ellipse: Plane cuts through one nappe at an angle, but does not intersect the base.
- Parabola: Plane is parallel to a generatrix (slant line) of the cone.
- Hyperbola: Plane intersects both nappes.

General Equation of a Conic Section

The set of all points (x, y) satisfying a quadratic polynomial equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A , B , and C are not all zero, is called a conic section. This algebraic definition encompasses all conic sections (circle, ellipse, parabola, and hyperbola). A conic is said to be degenerate if the polynomial factors into two linear terms (including possibly complex ones). It's worth noting that this definition is not exactly equivalent to the geometric definition as a plane section of a cone when restricted to real coordinates. For instance, the equation

$$x^2 + y^2 + 1 = 0$$

has no real solutions (an "imaginary" conic), yet in complex algebraic geometry, it corresponds to the intersection of $x^2 + y^2 = z^2$

with the plane $z = i$ which we cannot show on the real coordinate system[5, see further].

The term "conic section" originates from the cone-intersection definition, but expressing conics as second-degree polynomial equations often makes their analytical properties easier to study.

Polar Equation of a Conic. Using the eccentricity definition of a conic, the polar equation of a conic is given by

$$r = \frac{ep}{1 - e \cos \theta}$$

where p is the distance between the focus F and the directrix d , the pole of the polar axis is at F , and the polar axis is perpendicular to d and points away from d (see Fig. 2) [1, p. 469].

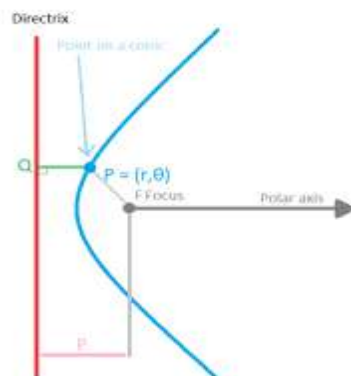


Figure 2: Polar representation

Circle

Definition. A circle is the set of all points in a plane that are at a fixed distance (radius) from a fixed point (center) (see Fig. 3).

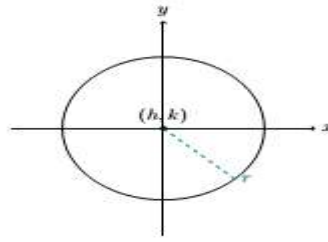


Figure 3: 2D representation

Standard Equation. For a circle with center (h, k) and radius r :

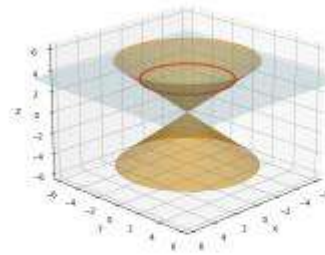
$$(x - h)^2 + (y - k)^2 = r^2$$


Figure 4: Conic representation from the code

Ellipse

Definition. An ellipse is the locus of points in a plane such that the sum of the distances from two fixed points (called the foci) is constant and greater than the distance between the foci. The constant sum of the distances of an arbitrary point on the ellipse from its foci is generally denoted by $2a$. The foci are designated F_1 and F_2 , and the distance between them is $2c$. By definition, $2a > 2c$, or equivalently, $a > c$ (see Fig. 5) [4, p. 83–84].

Standard Equation. If a rectangular Cartesian coordinate system is chosen so that the foci of the ellipse are symmetrically situated on the x -axis with respect to the origin, the equation of the ellipse is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b = \sqrt{a^2 - c^2}, a > b > 0. \quad (1)$$

This is called the canonical equation of the ellipse. Here, a is the semi-major axis, b is the semi-minor axis, and c is the distance from the center to each focus. (see Fig. 4 and Fig. 12 in [4, pp. 83–84]).

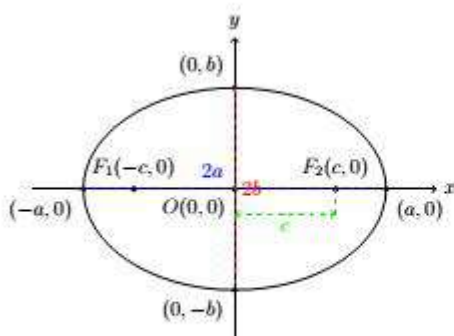


Figure 5: 2D representation

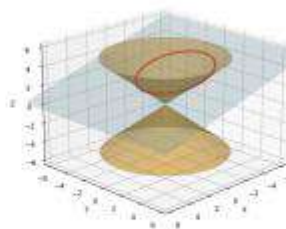


Figure 6: Conic representation from the code

Parabola.

Definition. A parabola is formed when a plane intersects one of the cones parallel to the generating line. A parabola can also be defined in terms of distances (see Fig. 7).

General Definition of Parabola: A parabola is the set of all points in a plane equidistant from a fixed point (focus) and a fixed line (directrix).

Standard Equation

$$y - k = a(x - h)^2 \text{ (opens upwards)}$$

$$x - h = a(y - k)^2 \text{ (opens sideways)}$$

Theorem: Given a parabola opening upward with vertex located at (h, k) and focus located at $(h, k + p)$, with constant p the equation for the parabola is:

$$y = \frac{1}{4p}(x - h)^2 + k \quad (2)$$

This is the standard form of a parabola [3, p. 75].

Explanation: The parabola shown is given by the equation $y = x^2$, which can be written in the standard form $y = \frac{1}{4p}x^2$, where $p = \frac{1}{4}$

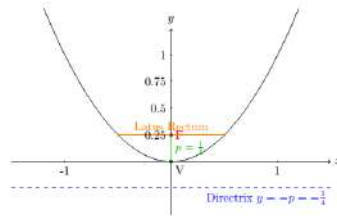


Figure 7: 2D representation

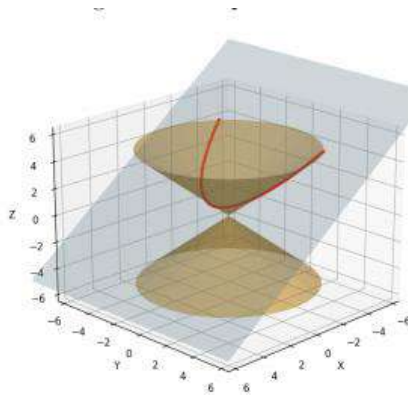


Figure 8: Conic representation from the code

Hyperbola

Definition. A hyperbola is the set of all points in a plane such that the absolute difference of the distances

from two fixed points (foci) is constant (see Fig. 9).

Standard Equation. Transverse axis along x :

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Transverse axis along y :

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

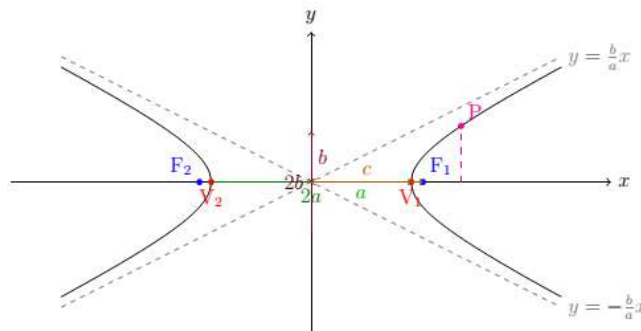


Figure 9: 2D representation

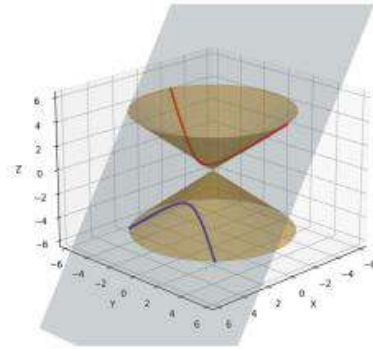


Figure 10: 3D conic representation from the code

Appendix A: Source Code for Interactive Conic Visualiser

```

1 # Import numerical and plotting libraries
2 import numpy as np                                # For mathematical
   operations and generating arrays
3 import matplotlib.pyplot as plt                    # For creating the 3D
   plot
4 from matplotlib.backends.backend_tkagg import FigureCanvasTkAgg #
   Embeds matplotlib inside Tkinter
5 # Import GUI toolkit
6 import tkinter as tk                               # Core Tkinter GUI
   framework
7 from tkinter import ttk                             # Themed Tkinter widgets
8

```

```

9 # Main application class
10 class ConicSectionApp(tk.Tk):
11     def __init__(self):
12         super().__init__()
13
14         # Window setup
15         self.title("Double Cone with Cutting Plane - Interactive")
16         self.geometry("1100x750")
17
18         # Variables controlling the cone, plane, and view
19         self.height = tk.DoubleVar(value=5)          # Height of
20             each half of the double cone
21         self.radius = tk.DoubleVar(value=5)          # Base radius
22         self.plane_angle = tk.IntVar(value=45)       # Angle of
23             cutting plane
24         self.plane_intercept = tk.DoubleVar(value=0) # Vertical
25             shift of plane
26         self.view_angle = tk.StringVar(value="Diagonal")
27         self.show_plane = tk.BooleanVar(value=True)
28         self.show_cone = tk.BooleanVar(value=True)
29         self.show_intersection = tk.BooleanVar(value=True)
30         self.auto_rotate = tk.BooleanVar(value=False)
31         self.azimuth = 45                           # Camera
32             rotation angle
33
34         # IDs for scheduled callbacks
35         self._update_id = None
36         self._rotate_id = None
37
38         # Build UI and draw initial figure
39         self.create_widgets()
40         self.plot_cone_and_plane()
41
42         # Create all GUI components on the left panel
43     def create_widgets(self):
44         control_frame = ttk.Frame(self)

```

```

41     control_frame.pack(side=tk.LEFT, fill=tk.Y, padx=10, pady
42         =10)
43
44     # Section title
45     ttk.Label(control_frame, text="Parameters", font=("Arial",
46         11, "bold")).pack(pady=(0, 10))
47
48     # Sliders for each parameter
49     self.add_slider(control_frame, "Cone height", self.height,
50         2, 10, resolution=0.5)
51     self.add_slider(control_frame, "Cone base radius", self.
52         radius, 2, 10, resolution=0.5)
53     self.add_slider(control_frame, "Plane angle (deg)", self.
54         plane_angle, -89, 89, resolution=1)
55     self.add_slider(control_frame, "Plane intercept", self.
56         plane_intercept, -10, 10, resolution=0.5)
57
58     # Visibility toggles
59     ttk.Separator(control_frame, orient='horizontal').pack(fill=
60         'x', pady=15)
61     ttk.Label(control_frame, text="Visibility", font=("Arial",
62         11, "bold")).pack(pady=(0, 5))
63
64     ttk.Checkbutton(control_frame, text="Show Cone", variable=
65         self.show_cone,
66         command=self.plot_cone_and_plane).pack(
67             anchor=tk.W, padx=5)
68
69     ttk.Checkbutton(control_frame, text="Show Plane", variable=
70         self.show_plane,
71         command=self.plot_cone_and_plane).pack(
72             anchor=tk.W, padx=5)
73
74     ttk.Checkbutton(control_frame, text="Show Intersection",
75         variable=self.show_intersection,
76         command=self.plot_cone_and_plane).pack(
77             anchor=tk.W, padx=5)
78
79     # View controls
80     ttk.Separator(control_frame, orient='horizontal').pack(fill=
81         'x', pady=15)
82     ttk.Label(control_frame, text="View Controls", font=("Arial"
83         , 11, "bold")).pack(pady=(0, 5))
84
85     for v in ["Front", "Side", "Top", "Diagonal"]:
86         ttk.Radiobutton(control_frame, text=v, value=v, variable
87             =self.view_angle,
88             command=self.schedule_update).pack(
89                 anchor=tk.W, padx=5)

```

```

73     # Auto rotation toggle
74     ttk.Checkbutton(control_frame, text="Auto-rotate", variable=
75         self.auto_rotate,
76         command=self.toggle_rotation).pack(anchor=tk
77             .W, padx=5, pady=(5, 0))
78
79     # Buttons for reset and update
80     ttk.Separator(control_frame, orient='horizontal').pack(fill=
81         'x', pady=15)
82     ttk.Button(control_frame, text="Reset View", command=self.
83         reset_view).pack(fill='x', pady=2)
84     ttk.Button(control_frame, text="Update Plot", command=self.
85         plot_cone_and_plane).pack(fill='x', pady=2)
86
87     # Preset conic sections (adjust plane angle & intercept)
88     ttk.Separator(control_frame, orient='horizontal').pack(fill=
89         'x', pady=15)
90     ttk.Label(control_frame, text="Presets", font=("Arial", 11,
91         "bold")).pack(pady=(0, 5))
92
93     ttk.Button(control_frame, text="Circle", command=lambda:
94         self.load_preset(0, 3)).pack(fill='x', pady=2)
95     ttk.Button(control_frame, text="Ellipse", command=lambda:
96         self.load_preset(15, 1)).pack(fill='x', pady=2)
97     ttk.Button(control_frame, text="Parabola", command=lambda:
98         self.load_preset(26, 2)).pack(fill='x', pady=2)
99     ttk.Button(control_frame, text="Hyperbola", command=lambda:
100         self.load_preset(50, 0)).pack(fill='x', pady=2)
101
102     # Create the Matplotlib figure area
103     self.fig = plt.figure(figsize=(8, 7))          # Full plot
104     window
105     self.ax = self.fig.add_subplot(111, projection='3d') # 3D
106     axis
107     self.canvas = FigureCanvasTkAgg(self.fig, master=self) #
108     Embed in Tkinter
109     self.canvas.get_tk_widget().pack(side=tk.RIGHT, fill=tk.BOTH
110         , expand=True)
111
112     # Footer area
113     self.footer = ttk.Label(self, text="", anchor='center')
114
115     # Helper to create sliders with label
116     def add_slider(self, frame, text, variable, mn, mx, resolution
117         =1):
118         row = ttk.Frame(frame)
119         row.pack(fill='x', pady=3)
120         ttk.Label(row, text=text).pack(anchor=tk.W)
121         ttk.Scale(row, from_=mn, to=mx, variable=variable,

```

```

106         orient='horizontal', command=lambda e: self.
107             schedule_update(),
108             length=160).pack()
109
110 # Schedule plot update without redrawing instantly (avoids lag)
111 def schedule_update(self):
112     if self._update_id is not None:
113         self.after_cancel(self._update_id)
114         self._update_id = self.after(150, self.plot_cone_and_plane)
115
116 # Turn on/off auto rotation
117 def toggle_rotation(self):
118     if self.auto_rotate.get():
119         self.start_rotation()
120     else:
121         if self._rotate_id:
122             self.after_cancel(self._rotate_id)
123             self._rotate_id = None
124
125 # Automatically rotate the camera azimuth
126 def start_rotation(self):
127     self.azimuth = (self.azimuth + 2) % 360 # Move camera
128     angle
129     self.ax.view_init(elev=30, azim=self.azimuth)
130     self.canvas.draw()
131     self._rotate_id = self.after(80, self.start_rotation)
132
133 # Reset viewing angle to default
134 def reset_view(self):
135     self.azimuth = 45
136     self.view_angle.set("Diagonal")
137     self.plot_cone_and_plane()
138
139 # Load preset: (plane_angle, plane_intercept)
140 def load_preset(self, angle, intercept):
141     self.plane_angle.set(angle)
142     self.plane_intercept.set(intercept)
143     self.plot_cone_and_plane()
144
145 # Core drawing function: draws cone, plane, and intersection
146 def plot_cone_and_plane(self):
147     self.ax.clear()
148
149     h = self.height.get()
150     r = self.radius.get()
151
152     # Generate a double cone using radius = k * |z|
153     z = np.linspace(-h, h, 200)
154     theta = np.linspace(0, 2 * np.pi, 200)
155     Z, Theta = np.meshgrid(z, theta)

```

```

154     R = r * (np.abs(Z) / h)
155     X = R * np.cos(Theta)
156     Y = R * np.sin(Theta)
157
158     # Plot the cone surface
159     if self.show_cone.get():
160         self.ax.plot_surface(X, Y, Z, alpha=0.25, color='gray',
161                             edgecolor='none')
162
163     # Plane equation
164     a = np.tan(np.radians(self.plane_angle.get()))
165     b = self.plane_intercept.get()
166
167     # Plane mesh
168     px = np.linspace(-r, r, 50)
169     py = np.linspace(-r, r, 50)
170     PX, PY = np.meshgrid(px, py)
171     PZ = a * PX + b
172
173     # Plot plane
174     if self.show_plane.get():
175         self.ax.plot_surface(PX, PY, PZ, alpha=0.3, color='
176                             yellow')
177
178     # Intersection: solve cone eq = plane eq
179     mask = np.abs(PZ) <= h
180     if self.show_intersection.get():
181         self.ax.scatter(PX[mask], PY[mask], PZ[mask], s=1, color
182                         ='red')
183
184     # Apply view presets
185     view = self.view_angle.get()
186     if view == "Front":
187         self.ax.view_init(30, 0)
188     elif view == "Side":
189         self.ax.view_init(30, 90)
190     elif view == "Top":
191         self.ax.view_init(90, 0)
192     else:
193         self.ax.view_init(30, self.azimuth)
194
195     self.ax.set_xlabel("X")
196     self.ax.set_ylabel("Y")
197     self.ax.set_zlabel("Z")
198     self.canvas.draw()
199
200 # Launch application
201 if __name__ == "__main__":
202     app = ConicSectionApp()
203     app.mainloop()

```

Conclusion. This document covered the four fundamental conic sections, the circle, the ellipse, the parabola, and the hyperbola, along with their definitions, equations, and visual representations from interactive code. Although all of them originate from different ways of cutting a cone, each curve has its own unique behaviour, formulae, and set of properties.

What makes these shapes particularly fascinating is how frequently they appear outside of pure mathematics. From planetary orbits and satellite trajectories to architectural designs and various applied fields, conic sections play a vital role in shaping our understanding of the world.

MATHEMATICS

By combining analytic geometry, polar and Cartesian algebra, and real-world examples, this document highlights not only what conic sections are, but also why they remain essential in science, engineering, and everyday problem-solving.

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ЗАДАЧА ТИПА СТЕФАНА ДЛЯ ДИФFUЗИОННОЙ СИСТЕМЫ ЛОТКИ-ВОЛЬТЕРРЫ

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Аннотация: В статье исследуется задача со свободной границей для диффузионной конкурирующей системы Лотки–Вольтерры с двумя подвижными границами, которые описывают фронты распространения двух конкурирующих видов. Модель включает нелинейные конвективные члены вида m_1uu_x и m_1vv_x , что отличает её от классических систем реакции-диффузии. Свободные границы удовлетворяют условиям типа Стефана. При соответствующих начальных и граничных условиях доказаны существование и единственность классического решения с использованием оценок Шаудера, принципа Лерэ–Шаудера и интегральных представлений. Получены априорные оценки и проанализирована дихотомия «распространение–исчезновение». В частности, показано, что если предел распространения конечен, то соответствующая плотность вида стремится к нулю. Результаты обобщают известные работы по задачам со свободной границей для одного и двух видов на случай нелинейной конвекции и двух различных фронтов.

Ключевые слова: Задача Стефана; свободная граница; система Лотки–Вольтерры; реакция-диффузия; нелинейная конвекция; дихотомия распространение–исчезновение; существование; единственность; асимптотическое поведение; конкуренция двух видов.

ON A STEFAN TYPE PROBLEM FOR A DIFFUSIVE LOTKA-VOLTERRA SYSTEM

Abstract: This paper studies a free boundary problem for a diffusive Lotka–Volterra competition system with two moving boundaries, where each boundary represents the spreading front of a competing species. The model includes nonlinear convection terms of the form m_1uu_x and m_1vv_x , which distinguishes it from classical reaction–diffusion systems. The free boundaries satisfy Stefan-type conditions. Under appropriate initial and boundary conditions, the existence and uniqueness of a classical solution are proved using Schauder-type estimates, the Leray–Schauder principle, and integral representations. A priori estimates are derived, and the spreading–vanishing dichotomy is analyzed. In particular, it is shown that if the spreading limit is finite, then the corresponding species density tends to zero. The results extend earlier work on free boundary problems for single-species and two-species models by incorporating nonlinear convection and two distinct fronts.

Keywords: Stefan problem; free boundary; Lotka–Volterra system; reaction–diffusion; nonlinear convection; spreading–vanishing dichotomy; existence; uniqueness; asymptotic behavior; two-species competition.

LOTKA-VOLTERRA DIFFUZION SISTEMASI UCHUN STEFAN MASALASI

Annotatsiya: Ushbu maqolada diffuziv Lotka – Volterra raqobat tizimi uchun ikki harakatlanuvchi chegarali erkin chegara masalasi o'rganiladi. Har bir chegara raqobatlashuvchi turlarning tarqalish frontini ifodalaydi. Model m_1uu_x va m_1vv_x ko'rinishidagi chiziqli konveksiya hadlarini o'z ichiga oladi, bu uni klassik reaksiya-diffuziya tizimlaridan farq qiladi. Erkin chegaralar Stefan tipidagi shartlarni qanoatlantiradi. Tegishli boshlang'ich va chegaraviy shartlar ostida klassik yechimning mavjudligi va yagonaligi Shauder tipidagi baholar, Lere – Shauder prinsipi va integral ifodalar yordamida isbotlangan. Aprior baholar olingan va “tarqalish –yo'qolish” dioxotomiyasi tahlil qilingan. Xususan, agar tarqalish chegarasi chekli bo'lsa, u holda tur zichligi nolga intilishi ko'rsatilgan. Natijalar bir va ikki turli modellar uchun erkin chegara masalalari bo'yicha oldingi ishlarni chiziqli konveksiya va ikki xil front holatiga umumlashtiradi.

Kalit so‘zlar: Stefan masalasi; erkin chegara; Lotka – Volterra tizimi; reaksiya-diffuziya; nochiziqli konveksiya; tarqalish – yo‘qolish dixotomiyasi; mavjudlik; yagonalik; asimptotik harakat; ikki turning raqobati.

Введение. Распространение новых или инвазивных видов является центральной темой экологии, и значительные исследования были посвящены лучшему пониманию природы такого распространения. Экологические проблемы требуют использования целой иерархии моделей, способных описывать не только разные уровни организации систем, но и взаимодействие между этими уровнями. Тем не менее, значительные успехи были достигнуты в исследованиях инвазии видов посредством исследования фронтального распространения (см., напр, [4, 8, 12, 16, 17]).

Подход, предложенный в работе [7], моделирует явление распространения для одного вида, принимая фронт распространения как свободную границу, где ключевым предположением является то, что плотность популяции исчезает на фронте, а механизм распространения - определяется пространственным градиентом популяции на фронте.

Следуя такому подходу, существуют различные биологические соображения в отношении двухвидовых моделей конкуренции. В работе [8] авторы считают, что инвазивный вид изначально существует в некотором промежутке и вторгается в окружающую среду, тогда как резидентный вид распространяется по всему пространству. В работах [10, 19] предполагается, что два вида со слабой конкуренцией распространяются вдоль одной и той же свободной границы. Аналогичные работы, но для двухвидовых моделей хищник – жертва, можно найти в [20]. Мы также ссылаемся на гораздо более ранние работы [5, 13], в которых среда считается ограниченной областью.

Во многих исследованиях термин конвекция является линейным и зависит только от градиента плотности компонентов [9, 15]. Однако в целом на конвекцию также влияет плотность компонентов, что, в свою очередь, приводит к нелинейной конвекции. В [18] авторы исследовали задачу со свободной границей для уравнения реакции-диффузии с нелинейным членом конвекции. Они получили результат дихотомии и представили постоянную асимптотическую скорость распространения расширяющегося фронта.

На основании этих работ мы можем уточнить: если два вида $u(t, x)$ и $v(t, x)$ распространяются только в одном направлении, но с разными свободными границами, то какова может быть динамика. Точнее, мы предполагаем, что два вида изначально занимают интервалы $[0, s_1(t)]$ и $[0, s_2(t)]$ соответственно. Кроме того, они перемещаются только вправо, и их территория расширяется до $[0, s_1(t)]$ и $[0, s_2(t)]$, соответственно, в момент времени t .

В этой статье мы изучаем динамику конкурентной системы типа Лотки – Вольтерра с двумя свободными границами, введенными в [11, 21], где две свободные границы используются для описания фронтов распространения двух конкурирующих видов, соответственно. Взаимодействие между двумя конкурирующими видами формулируется как следующая задача со свободной границей:

$$u_t - d_1 u_{xx} - m_1 u u_x = f(u, v) \equiv u(a_1 - b_1 u - c_1 v), \quad (t, x) \in D, \quad (1.1)$$

$$v_t - d_2 v_{xx} - m_2 v v_x = g(u, v) \equiv v(a_2 - b_2 v - c_2 u), \quad (t, x) \in Q, \quad (1.2)$$

$$u(0, x) = u_0(x), \quad 0 \leq x \leq s_0, \quad (1.3)$$

$$v(0, x) = v_0(x), \quad 0 \leq x \leq h_0, \quad (1.4)$$

$$u_x(t, 0) = 0, \quad v_x(t, 0) = 0, \quad 0 \leq t \leq T, \quad (1.5)$$

$$u(t, s(t)) = 0, \quad v(t, h(t)) = 0, \quad 0 \leq t \leq T, \quad (1.6)$$

$$s'(t) = -\mu_1 u_x(t, s(t)), \quad 0 \leq t \leq T, \quad (1.7)$$

$$h'(t) = -\mu_2 v_x(t, h(t)), \quad 0 \leq t \leq T, \quad (1.8)$$

$$s(0) = s_0, \quad h(0) = h_0, \quad s_0 < h_0; \quad u(t, x) \equiv 0, \quad s(t) \leq x \leq h(t), \quad (1.9)$$

где $D = \{(t, x): 0 < t \leq T, 0 < x < s(t)\}$, $Q = \{(t, x): 0 < t \leq T, 0 < x < h(t)\}$; $u(t, x)$, $v(t, x)$ – плотности популяции в момент времени t в точке x ; $s(t)$, $h(t)$ – свободные границы, которые представляют фронты распространения, определяются вместе с функциями $u(t, x)$, $v(t, x)$; d_i – коэффициент диффузии, a_i – коэффициенты удельной скорости роста популяций, $i = 1, 2$; c_1 и b_2 – коэффициенты межвидовой конкуренции, c_2 и b_1 – коэффициенты внутривидовой конкуренции, $u_0(x)$, $v_0(x)$ – начальные плотности популяции соответственно находятся в областях $[0, s_0]$ и $[0, h_0]$.

А параметры μ_i соответственно представляют скорость распространения в новые области для $u(t, x)$ и $v(t, x)$.

Относительно данных задачи предполагаются выполненными следующие условия:

- $d_i, m_i, a_i, b_i, c_i, \mu_i$ – положительные постоянные, $i = 1, 2$;
- $u_0(x), v_0(x) \in C^2[0, s_0]$, $0 < u_0(x) \leq \frac{a_1}{b_1}$ в $(0, s_0)$, $0 < v_0(x) \leq \frac{a_2}{c_2}$ в $(0, h_0)$,
- $u_0(0) = v_0(0) = 0, u_0(s_0) = v_0(h_0) = 0, u_0'(s_0) < 0, v_0'(h_0) < 0$,
- $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$.

Задача (1.1)–(1.9) исследована в работе [11, 21] при $m_i \equiv 0, i = 1, 2$.

Априорные оценки. В этом разделе установим некоторые априорные оценки шаудеровского типа, которые будут использованы при доказательстве глобальной разрешимости задачи. При этом широко применяются принцип максимума и теоремы сравнений.

Прежде всего установим ограниченность функции $u(t, x), v(t, x), s'(t), h'(t)$. Далее устанавливаются априорные оценки для старших производных.

Теорема 2.1. Если функции $u(t, x), v(t, x), s(t), h(t)$ являются решением задачи (1.1)–(1.9), то справедливы оценки

$$0 < u(t, x) \leq M_1 \equiv \frac{a_1}{b_1}, \quad (t, x) \in \bar{D}, \quad (2.1)$$

$$0 < v(t, x) \leq M_2 \equiv \frac{a_2}{c_2}, \quad (t, x) \in \bar{Q}, \quad (2.2)$$

$$0 < s'(t) \leq M_3 \equiv \mu_1 N_1, \quad 0 \leq t \leq T, \quad (2.3)$$

$$0 < h'(t) \leq M_4 \equiv \mu_2 N_2, \quad 0 \leq t \leq T, \quad (2.4)$$

$$\text{где } N_1 \geq \max \left\{ \frac{a_1}{m_1}, \max_x \left| \frac{u_0}{s_0 - x} \right| \right\}, \quad N_2 \geq \max \left\{ \frac{a_2}{m_2}, \max_x \left| \frac{v_0}{h_0 - x} \right| \right\}.$$

Доказательство. Сначала докажем положительность функции $u(t, x)$. Возьмем некоторую произвольную точку $P \in D$ такую, что $u(P) = 0$. В этой точке правая часть (1.1) должна быть равна нулю. А также в этой точке функция $u(t, x)$ достигает своего минимального значения. Отсюда по сильному принципу максимума $u(P) = u(P_0)$ для всех $P_0 \in D_0 = \{(t_0, x): 0 < t_0 < T, 0 < x < s(t_0)\}$ и получаем противоречие относительно условия (ii). Полученное противоречие доказывает, что $u(t, x) > 0$ в D .

Доказательство положительности $v(t, x)$ аналогично.

Чтобы получить верхние оценки поступаем следующим образом. Построим вспомогательную задачу для обыкновенного дифференциального уравнения:

$$\begin{cases} \bar{u}_t(t) = \bar{u}(a_1 - b_1 \bar{u}), & t > 0, \\ \bar{u}(0) = \|u_0\|_\infty \end{cases}$$

Её решение дается следующей явной формулой:

$$\bar{u}(t) = \frac{a_1}{b_1} e^{\frac{a_1}{b_1} t} \left(e^{-\frac{a_1}{b_1} t} - 1 + \frac{a_1}{b_1 \|u_0\|_\infty} \right)^{-1}.$$

Используя лемму 4.1, сравнивая $u(t, x)$ с $\bar{u}(t)$ получим

$$u(t, x) \leq \sup_{t \geq 0} \bar{u}(t) \equiv M_1.$$

Далее, с учетом условия (1.6) и положительности функции $u(t, x)$ в области D , находим $u_x(t, s(t)) < 0$. Следовательно, из (1.7) получим $s'(t) > 0$.

Для того, чтобы установить верхнюю оценку для $s'(t)$, в задаче (1.1), (1.3), (1.5), (1.6), производя замену

$$U(t, x) = u(t, x) + N_1(x - s(t))$$

и получаем задачу относительно $U(t, x)$

$$\begin{cases} U_t - d_1 U_{xx} - m_1 U_x \leq u(a_1 - m_1 N_1) \leq 0, & (t, x) \in D, \\ U(0, x) = u_0(x) + N_1(x - s_0) \leq 0, & 0 \leq x \leq s_0, \\ U_x(t, 0) = N_1, \quad U(t, s_1(t)) = 0, & 0 \leq t \leq T. \end{cases}$$

За счет выбора $N_1 \geq \max \left\{ \frac{a_1}{m_1}, \max_x \left| \frac{u_0}{s_0 - x} \right| \right\}$ всюду в D имеем $U(t, x) \leq 0$. Следовательно, с учетом (2.5) находим $u_x(t, s(t)) \geq -N_1$, что эквивалентно $s'(t) \leq \mu_1 N_1$.

Рассуждая аналогичным образом, мы получим оценки (2.2) и (2.4). $\square$

Первоначальные оценки получены. Теперь, используя результаты работы [1], установим априорные оценки производных u_x и v_x .

Для каждого уравнения системы отдельно сформулируем соответствующую задачу:

$$u_t - d_1 u_{xx} - m_1 u u_x = f(u, v), \quad (t, x) \in D_1, \quad (2.6)$$

$$u(0, x) = u_0(x), \quad 0 \leq x \leq s_0, \quad (2.7)$$

$$u_x(t, 0) = 0, \quad u(t, s(t)) = 0, \quad 0 \leq t \leq T, \quad (2.8)$$

$$s'(t) = -\mu_1 u_x(t, s(t)), \quad 0 \leq t \leq T, \quad (2.9)$$

и

$$v_t - d_2 v_{xx} - m_2 u v_x = g(u, v), \quad (t, x) \in D_2, \quad (2.10)$$

$$v(0, x) = v_0(x), \quad 0 \leq x \leq h_0, \quad (2.11)$$

$$v_x(t, 0) = 0, \quad v(t, h(t)) = 0, \quad 0 \leq t \leq T, \quad (2.12)$$

$$h'(t) = -\mu_2 v_x(t, h(t)), \quad 0 \leq t \leq T, \quad (2.13)$$

Сначала получим априорные оценки для $u(t, x)$. В силу граничных условий (2.6)–(2.9) мы не можем использовать результаты работы [1]. Поэтому введем следующее преобразование

$$t = t, \quad y = \frac{x}{s(t)}$$

чтобы выпрямить свободную границу. Тогда область D преобразовывается в область $D_1 = \{(t, y): 0 < t < T, 0 < y < 1\}$, а ограниченная функция $U(t, y) = u(t, x)$ является решением задачи

$$U_t - A_1 U_{yy} = B_1(U, V, U_y), \quad (t, y) \in D_1, \quad (2.14)$$

$$U(0, y) = U_0(y) \equiv u_0(s_0 y), \quad 0 \leq y \leq 1, \quad (2.15)$$

$$U_y(t, 0) = 0, \quad U(t, 1) = 0, \quad 0 \leq t \leq T, \quad (2.16)$$

где $D_1 = \{(t, y): 0 < t \leq T, 0 < y < 1\}$, $A_1 = \frac{d_1}{s^2(t)}$, $B_1(U, V, U_y) = f(U, V) + \frac{y s'(t) + m_1 U}{s(t)} U_y$, $s'(t) = -\frac{\mu_1}{s(t)} U_y(t, s(t))$.

Уравнение (2.14) удовлетворяет условиям теоремы 3 работы [1]:

$$\frac{|B_1|}{A_1} \leq K_1 (U_y^2 + 1), \quad K_1 > 0.$$

Следовательно имеет место:

Теорема 2.2. В условиях (2.17) справедливо оценка

$$|U_y(t, y)| \leq M_5(M_1, M_2, A_1, K_2, \delta), \quad (t, y) \in D_1^\delta.$$

Кроме того, в области $\{(t, y) \in \bar{D}_1, |U| \leq M_1, |V| \leq M_2, |U_y| \leq M_5\}$ имеет место оценка

$$|U|_{\frac{2}{3}}^{D_1^{2\delta}} \leq M_6(M_1, M_2, \frac{d_1}{s_0^2}, K_2, \delta).$$

И если еще известно, что $U(t, y)$ обладает в $\bar{D}_1$ суммируемыми с квадратом обобщенными производными U_{ty}, U_{yy} , то

$$|U|_{1+\alpha_1}^{D_1^{2\delta}} \leq M_7(M_1, M_2, A_{01}, M_5, K_2, \delta), \quad 0 < \alpha_1 < 1,$$

Если $U|_{\Gamma(t=0, y=0, y=1)} = 0$, то оценки (2.12)–(2.14) справедливы и в $\bar{D}_1$; где $\Gamma(t=0, y=0, y=1)$ – параболическая граница, $D_1^\delta = \{(t, y): 0 < \delta \leq t \leq T, \delta \leq y \leq 1 - \delta\}$.

Доказательство. Так как установлены оценки $|u(t, x)| \leq M_1$, $|v(t, x)| \leq M_2$, $|s_1(t)| \leq M_3$, $|s_2(t)| \leq M_4$, соответственно получается ограниченности функции $U(t, y)$ и $V(t, y)$, то в силу теоремы 3 работы [1] справедливы внутренние оценки (2.18), (2.19).

Если $U(t, 1) \equiv 0$, то доопределяем функцию $U(t, y)$ в прямоугольнике $R_+ = \{(t, y): 0 \leq t \leq T, 1 - 4\delta < y < 1 + 4\delta\}$, $0 < \delta < \frac{1}{5}$ по формуле $U(t, y) = -W(t, 1 - 4\delta(2y - 1))$, $V(t, y) = V(t, y)$. Построенная в прямоугольнике $D_1^\delta \cup R_+$ функция (за которой сохраним обозначение $U(t, y)$) непрерывна вместе с производной их и всюду, кроме точек прямой $y = 1$, удовлетворяет уравнению вида (2.14), для которого выполнены условия (2.15) и (2.16):

$$\begin{cases} W_t = 64\delta A_1 W_{yy} + B_1(t, y, -W, W_y, V), & (t, y) \in R_+, \\ W(\delta, y) = W_0(y), & 1 - 4\delta \leq y \leq 1 + 4\delta, \\ W(t, 1 - 4\delta) = 0, \quad W(t, 1 + 4\delta) = 0, & \delta \leq t \leq T. \end{cases}$$

Теперь применяя теорему 2 [1] к задаче (2.20) получим оценку $|W_y| \leq M_5(M_1, M_2, A_1, K_2, \delta)$ в области R_+ . Следовательно,

$$|U_y| \leq M_5(M_1, M_2, A_1, K_2, \delta) \quad (t, y) \in R_+.$$

Аналогично доопределяем функцию $U(t, y)$ в прямоугольнике $R_- = \{(t, y): 0 \leq t \leq T, -4\delta < y < 4\delta\}$ по формуле $U(t, y) = W(t, 4\delta(1 - 2y))$, $V(t, y) = V(t, y)$.

Оценки в области R_+, R_-, D_1^δ дают общую оценку $|U_y| \leq M_4(M_1, M_2)$ в замкнутой области $\bar{D}_1$.

Далее, используя теорему 3 [1] получим оценка $|U|_{1+\alpha_1} \leq M_7(M_1, M_2, A_{01}, M_5, K_2)$ в $\bar{D}_1$.

Оценки старших производных в $\bar{D}_1$ устанавливаются при помощи результатов для линейных уравнений [6].

Теорема 2.3. Пусть коэффициенты уравнения

$$a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u - u_t = f(t, x), \quad (t, x) \in \bar{D}_1$$

удовлетворяет условиям Гёльдера

$$a(t, x) \geq a_0 > 0, \quad |a|_{\bar{D}_1} + |b|_{\bar{D}_1} + |c|_{\bar{D}_1} = K < \infty,$$

Пусть $u(t, x)$ – решение уравнения (2.21), $|u|_{2+\gamma}^{\bar{D}_1} < \infty$, $M_1 = \max|u(t, x)|$. Тогда

$$|u|_{2+\beta_1}^{D_1^{4\delta}} \leq C(|f|_{\bar{D}_1} + M)$$

где C зависят от β_1, K, δ .

Если $u|_\Gamma = 0$, то существует такое $C = C(\beta_1, a_0, K)$, что $|u|_{2+\beta_1}^{D_1^{4\delta}} \leq C|f|_{\bar{D}_1}$.

В случае задачи (2.10)–(2.13) априорные оценки строятся следующим образом. Произведя замену $y = \frac{x}{h(t)}$, распрямляем границу. Тогда область Q переходит в область $Q_1 = \{(t, y): 0 < t \leq T, 0 < y < 1\}$, а для функции $V(t, y) = v(t, x)$ получается задача для уравнении с ограниченными коэффициентами и правой частью:

$$V_t - A_2 V_{yy} = B_2(U, V, U_y), \quad (t, y) \in Q_1, \quad (2.22)$$

$$V(0, y) = V_0(y) \equiv v_0(h_0 y), \quad 0 \leq y \leq 1, \quad (2.23)$$

$$V_y(t, 0) = 0, \quad V(t, 1) = 0, \quad 0 \leq t \leq T, \quad (2.24)$$

где $Q_1 = \{(t, y): 0 < t \leq T, 0 < y < 1\}$, $A_2 = \frac{d_2}{h(t)}$, $B_2(U, V, U_y) = g(U, V) + \frac{yh'(t) + m_2 V}{h(t)} V_y$,

$$h'(t) = -\frac{\mu_2}{h(t)} V_y(t, h(t)).$$

Рассуждая так же, как и выше используя методы работы [1], установим оценки для $|V_y|, |V|_{1+\alpha_2}, |V|_{2+\beta_2}$ в Q_1 . $\square$

Единственность и существование решения. Для доказательства единственности решения выводим интегральное представление, эквивалентное к (1.8) и (1.9)

$$s(t) = s_0 + \int_0^{s_0} u_0(\xi) d\xi - \int_0^{s(t)} u(t, \xi) d\xi + \frac{1}{2} \int_0^t u^2(t, \xi) d\xi - \int_0^t d\eta \int_0^{s(\eta)} f(u, v) d\xi$$

и

$$h(t) = h_0 + \int_0^{h_0} v_0(\xi) d\xi - \int_0^{h(t)} v(t, \xi) d\xi + \frac{1}{2} \int_0^t v^2(t, \xi) d\xi - \int_0^t d\eta \int_0^{h(\eta)} g(u, v) d\xi$$

Для простоты рассмотрена, случай $d_i = m_i = a_i = b_i = \mu_i = 1, i = 1, 2$.

Теорема 3.1. При выполнении условий теоремы 2.1 решение задачи (1.1)–(1.9) единственно.

Доказательство. Теорема доказывается сначала для малого $t \in [0, t_0]$, а затем устанавливается, что она справедлива для любого $0 < t < \infty$. Пусть функции $s_1(t), s_2(t), u_1(t, x), v_1(t, x)$ и $h_1(t), h_2(t), u_2(t, x), v_2(t, x)$ являются решениями задачи (1.1)–(1.9) и пусть $y_i(t) = \min(s_i(t), h_i(t)), z_i(t) = \max(s_i(t), h_i(t))$. Для каждой группы решений справедливо (3.1), (3.2). Рассматривая разность, находим

$$|s_1(t) - s_2(t)| \leq \int_0^{y_1(t)} |u_1(t, \xi) - u_2(t, \xi)| d\xi + \frac{1}{2} \int_0^t (u_1(\eta, 0) + u_2(\eta, 0)) |u_1(\eta, 0) - u_2(\eta, 0)| d\eta +$$

$$+ \int_{y_1(t)}^{z_1(t)} |u_i(t, \xi)| d\xi + \int_0^t d\eta \int_0^{y_1(\eta)} |f(u_1, v_1) - f(u_2, v_2)| d\xi + \int_0^t d\eta \int_{y_1(\eta)}^{z_1(\eta)} |f(u_i, v_i)| d\xi$$

и

$$|h_1(t) - h_2(t)| \leq \int_0^{y_2(t)} |v_1(t, \xi) - v_2(t, \xi)| d\xi + \frac{1}{2} \int_0^t (v_1(\eta, 0) + v_2(\eta, 0)) |v_1(\eta, 0) - v_2(\eta, 0)| d\eta +$$

$$+ \int_{y_2(t)}^{z_2(t)} |v_i(t, \xi)| d\xi + \int_0^t d\eta \int_0^{y_2(\eta)} |g(u_1, v_1) - g(u_2, v_2)| d\xi + \int_0^t d\eta \int_{y_2(\eta)}^{z_2(\eta)} |g(u_i, v_i)| d\xi$$

где u_i, v_i ($i = 1, 2$) – решения между $y_i(t)$ и $z_i(t)$, т.е

$$u_i(t, x) = \begin{cases} u_1(t, x), & \text{если } s_2(t) < s_1(t), \\ u_2(t, x), & \text{если } s_2(t) > s_1(t). \end{cases}$$

$$v_i(t, x) = \begin{cases} v_1(t, x), & \text{если } h_2(t) < h_1(t), \\ v_2(t, x), & \text{если } h_2(t) > h_1(t). \end{cases}$$

В силу теоремы 2.1 имеем

$$|u_i(t, x)| \leq N_1(y_1(t) - x), \quad |v_i(t, x)| \leq N_2(y_2(t) - x),$$

$$|u_1(t, y_1(t)) - u_2(t, y_1(t))| \leq N_1 |s_1(t) - s_2(t)|,$$

$$|v_1(t, y_2(t)) - v_2(t, y_2(t))| \leq N_2 |h_1(t) - h_2(t)|.$$

Для разности $U(t, x) = u_1(t, x) - u_2(t, x), V(t, x) = v_1(t, x) - v_2(t, x)$ получаются задачи

$$\begin{cases} U_t - U_{xx} - m_1 U U_x - k_1 U = k_2 V, & 0 \leq t \leq t_0, 0 < x < y_1(t) \\ V_t - V_{xx} - m_2 V V_x - k_3 V = k_4 U, & 0 \leq t \leq t_0, 0 < x < y_2(t), \\ U(0, x) = 0, & 0 \leq x \leq y_1(0), \\ V(0, x) = 0, & 0 \leq x \leq y_2(0), \\ U_x(t, 0) = 0, \quad V_x(t, 0) = 0, & 0 \leq t \leq t_0, \\ U(t, y_1(t)) \leq N_1 \max_{0 \leq \eta \leq t} |s_1(\eta) - s_2(\eta)|, & 0 \leq t \leq t_0, \\ V(t, y_2(t)) \leq N_2 \max_{0 \leq \eta \leq t} |h_1(\eta) - h_2(\eta)|, & 0 \leq t \leq t_0, \end{cases}$$

где $k_1 = (1 - (u_1 + u_2) - v_2 + u_{1x}), k_2 = u_1, k_3 = (1 - (v_1 + v_2) - u_1 + v_{1x}), k_4 = v_2$.

Из задачи (3.5) по принципу максимума находим

$$|U(t, x)| \leq N_1 |s_1(t) - s_2(t)| + k_2 |V(t, x)| t, \quad 0 \leq t \leq t_0, 0 < x < y_1(t),$$

$$|V(t, x)| \leq N_2 |h_1(t) - h_2(t)| + k_4 |U(t, x)| t, \quad 0 \leq t \leq t_0, 0 < x < y_2(t).$$

Далее используя полученные результаты оцениваются интегральные члены из (3.3) и (3.4). Затем применяя методы работы [14] завершается доказательство теоремы 3.1. $\square$

Теорема 3.2. Пусть выполнены условия теоремы 2.1. Тогда существует решение $u(t, x) \in C^{2+\beta_1}(\bar{D}), v(t, x) \in C^{2+\beta_2}(\bar{Q}), s(t) \in C^{1+\alpha_1}(0 \leq t \leq T), h(t) \in C^{1+\alpha_2}(0 \leq t \leq T)$ задачи (1.1)–(1.9).

Доказательство. Для доказательства разрешимости нелинейной задачи можно использовать различные теоремы из теории нелинейных уравнений, учитывая, что для нее имеет место теорема единственности классического решения. Мы применим принцип Лерэ–Шаудера [2], установленные априорные оценки $|\cdot|_{1+\alpha_i}$ для всех возможных решений нелинейных задач и теоремы разрешимости в классах Гёльдера для линейных задач. При этом теоремы существования для систем такие же, как и теорема для случая одного уравнения, так как каждое из уравнений можно рассматривать как линейное уравнение относительно $u(t, x)$ и $v(t, x)$ с непрерывными по Гёльдеру коэффициентами.

Задача (1.1)–(1.9) рассматривается одновременно с однопараметрическим семейством задач того же типа. Линейная задача определяет преобразование $w = F(\omega, k), 0 \leq k \leq 1$, к которому и применяется принцип Лерэ–Шаудера. Этот оператор нелинеен и зависит от k . Его неподвижные точки при $k = 1$ являются решениями задачи.

Обозначим через $H^{1+\alpha_i}, \alpha_i \in (0, 1)$ банахово пространство функций $u(t, x), v(t, x)$ с нормой $\|\cdot\| = |\cdot|_{1+\alpha_i}$, которые удовлетворяют соответствующим начальным и краевым условиям задачи (2.6)–(2.9) и (2.10)–(2.13).

Для каждой функции $\bar{u}, \bar{v} \in H^{1+\alpha_i}$ и любого числа $k \in [0, 1]$ обозначим через u^k, v^k решения линейных задач (2.6)–(2.9) и (2.10)–(2.13), решения которых существуют и единственны, причем $u^k, v^k \in C^{2+\beta_i}, s(t) \in C^{1+\alpha_1}, h(t) \in C^{1+\alpha_2}$. При этом в областях D и Q , осуществляется переход к параболическому уравнению с непрерывными по Гельдеру коэффициентами в фиксированной области.

Равномерная непрерывность и вполне непрерывность оператора преобразования F относительно k , равномерные по k оценки для решений и разрешимость линейных задач следуют из установленных априорных оценок норм Гельдера. Техника доказательства подробно продемонстрирована, например, в работах (гл. VI, [2]; гл. VII, [3]).

Этим и завершается доказательство теоремы. $\square$

Асимптотическое поведение. Сначала приведем некоторые сведения.

Определение 4.1. ([13] Гл. 8, п. 1). Пара функций (f, g) называется квазимоноотонной невозрастающей в $\mathbb{R}^2$, если $f(u, v)$ не возрастает по v для фиксированных $u \geq 0$ и $g(u, v)$ невозрастающей по u для фиксированных $v \geq 0$; это удовлетворяет $f(u, v) = u(a_1 - b_1 u - c_1 v)$, $g(u, v) = v(a_2 - b_2 u - c_2 v)$ в задаче (1.1)–(1.9) для $u \geq 0, v \geq 0$.

Нижние и верхние решения обозначим через $(\underline{u}, \underline{v})$ и $(\bar{u}, \bar{v})$, соответственно и они удовлетворяют

$$[(\underline{u}, \underline{v}), (\bar{u}, \bar{v})] = \{(u, v) \in [C([0, T] \times [0, \infty))]^2 : (\underline{u}, \underline{v}) \leq (u, v) \leq (\bar{u}, \bar{v})\}.$$

Лемма 4.1. (Принцип сравнения). Предположим, что $\bar{s}, \underline{h} \in C^1([0, T])$, $\bar{u} \in C(\bar{D}^*) \cap C^{1,2}(D^*)$ и $\underline{v} \in C(\bar{Q}^*) \cap C^{1,2}(Q^*)$ удовлетворяющие

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} - m_1 \bar{u} \bar{u}_x \geq f(\bar{u}, \underline{v}), & (t, x) \in D^*, \\ \underline{v}_t - d_2 \underline{v}_{xx} - m_2 \underline{v} \underline{v}_x \leq g(\bar{u}, \underline{v}), & (t, x) \in Q^*, \\ \bar{u}_x(t, 0) = \underline{v}_x(t, 0) = 0, & 0 < t \leq T, \\ \bar{u}(t, x) = 0, & 0 < t \leq T, \bar{s}(t) \leq x < \infty, \\ \underline{v}(t, x) = 0, & 0 < t \leq T, \underline{h}(t) \leq x < \infty, \\ \bar{s}'(t) \geq -\mu_1 \bar{u}_x(t, \bar{s}(t)), \underline{h}'(t) \leq -\mu_2 \underline{v}_x(t, \underline{h}(t)), & 0 < t \leq T. \end{cases}$$

где $D^* = \{(t, x) : t \in (0, T], x \in (0, \bar{s}(t))\}$, $Q^* = \{(t, x) : t \in (0, T], x \in (0, \underline{h}(t))\}$. Если $s_0 \leq \bar{s}(0)$, $h_0 \geq \underline{h}(0)$, $u_0(x) \leq \bar{u}(x, 0)$ для $x \in [0, s_0]$ и $v_0(x) \geq \underline{v}(x, 0)$ для $x \in [0, \underline{h}(0)]$, то для решений (u, v, s, h) задачи (4.1) выполняются

$$\begin{aligned} s(t) &\leq \bar{s}(t), \quad h(t) \leq \underline{h}(t), \quad 0 \leq t \leq T, \\ u(t, x) &\leq \bar{u}(t, x), \quad 0 \leq t \leq T, \quad 0 \leq x \leq \bar{s}(t), \\ v(t, x) &\geq \underline{v}(t, x), \quad 0 \leq t \leq T, \quad 0 \leq x \leq \underline{h}(t). \end{aligned}$$

Доказательство. Положим $M = \max_{t \in [0, T], x \in [0, \infty)} \{|v| + |\underline{v}|\}$, $w = M - v$ и $\bar{w} = M - \underline{v}$. То мы имеем

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} - m_1 \bar{u} \bar{u}_x \geq \bar{u}(a_1 - c_1 M - b_1 \bar{u} + c_1 \bar{w}), & (t, x) \in D^*, \\ \bar{w}_t - d_2 \bar{w}_{xx} - m_2 (M - \bar{w}) \bar{w}_x \geq (M - \bar{w})(a_2 - c_2 M - b_2 \bar{u} + c_1 \bar{w}), & (t, x) \in Q^*, \\ \bar{u}_x(t, 0) = \bar{w}_x(t, 0) = 0, & 0 \leq t \leq T, \\ \bar{u} \equiv 0 & 0 \leq t \leq T, \bar{s}(t) \leq x < \infty, \\ \bar{w} \equiv M & 0 \leq t \leq T, \underline{h}(t) \leq x < \infty, \\ \bar{s}'(t) \geq -\mu_1 \bar{u}_x(t, \bar{s}(t)), \underline{h}'(t) \leq \mu_2 \bar{w}_x(t, \underline{h}(t)) & 0 \leq t \leq T. \end{cases}$$

Теперь докажем, что

$$s(t) < \bar{s}(t), \quad \underline{h}(t) < h(t), \quad 0 < t \leq T.$$

Итак, пусть $s_0 < \bar{s}(0)$ и $\underline{h}(0) < h_0$. Тогда $s(t) < \bar{s}(t)$ и $\underline{h}(t) < h(t)$ для достаточно малых t . Предположим, что существует $t_0 \in (0, T]$ такое, что

$$s(t_0) = \bar{s}(t_0), \quad s(t) < \bar{s}(t), \quad \underline{h}(t) < h(t), \quad t \in (0, t_0)$$

или

$$\underline{h}(t_0) = h(t_0), \quad s(t) < \bar{s}(t), \quad \underline{h}(t) < h(t), \quad t \in (0, t_0)$$

Мы докажем случай (4.3). Поскольку $s(t) < \bar{s}(t)$, $\underline{h}(t) < h(t)$ при $t \in (0, t_0)$, то

$$\bar{s}'(t_0) \leq s'(t_0).$$

Из (4.4) и условия Стефана следует, что

$$u_x(t_0, s(t_0)) \leq \bar{u}_x(t_0, s(t_0)).$$

Теперь сравним u с $\bar{u}$ и w с $\bar{w}$. Сделаем замены $U(t, x) = (\bar{u}(t, x) - u(t, x))e^{-kt}$ и $W(t, x) = (\bar{w}(t, x) - w(t, x))e^{-kt}$. Для этих функций получим задачу

$$\begin{cases} \bar{U}_t - d_1 \bar{U}_{xx} - m_1 \bar{U} \bar{U}_x + (K_{11} - k)U + K_{12}V \geq 0, & 0 < t \leq t_0, \quad 0 < x < \bar{s}(t), \\ \bar{W}_t - d_2 \bar{W}_{xx} - m_2 \bar{W} \bar{W}_x + K_{21}U + (K_{22} - k)W \geq 0, & 0 < t \leq t_0, \quad 0 < x < \underline{h}(t), \\ U(0, x) \geq 0, \quad W(0, x) \geq 0, & x \geq 0, \\ \bar{U}_x(t, 0) = \bar{W}_x(t, 0) = 0, & 0 \leq t \leq t_0, \\ \bar{U} \equiv 0 & 0 \leq t \leq t_0, \quad \bar{s}(t) \leq x < \infty, \\ \bar{W} \geq 0 & 0 \leq t \leq t_0, \quad \underline{h}(t) \leq x < \infty. \end{cases}$$

где коэффициенты K_{ij} ($i, j = 1, 2$) — ограниченные функции, K_{12} и K_{21} неотрицательные функции.

Отсюда следует

$$U \geq 0, (t, x) \in D^*; \quad V \geq 0, (t, x) \in Q^*$$

при достаточно большом k .

Так как $U(t_0, s(t_0)) = 0$. Тогда в силу теоремы Хопфа должно быть $U_x(t_0, s(t_0)) < 0$ т.е. $u_x(t_0, s(t_0)) > \bar{u}_x(t_0, s(t_0))$ что противоречит (4.5). Таким образом, получаем (4.2). $\square$

Теорема 4.1. Пусть u, v, s, h является решением задачи (1.1)–(1.9). Если $s_\infty < \infty$ ($h_\infty < \infty$), то $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0$ ($\lim_{t \rightarrow \infty} \|v(t, \cdot)\|_{C([0, h(t)])} = 0$).

Доказательство. Мы докажем случай $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0$. Используя (1.1)–(1.9), имеем

$$u_t \geq d_1 u_{xx} + m_1 u u_x + u(a_1 - b_1 u - c_1 M_2), \quad t > 0, \quad 0 < x < s(t).$$

Предполагаем, что существуют $\sigma > 0$ и $\{(t_j, x_j)\}_{j=1}^\infty$, как $0 \leq x_j \leq s(t_j)$ и $t_j \rightarrow \infty$ при $j \rightarrow \infty$, такое что

$$u(t_j, x_j) \geq 4\sigma, \quad j = 1, 2, \dots$$

Так как $0 \leq x_j \leq s_\infty$, существует подпоследовательность $\{x_j\}$ и $x_0 \in [0, s_\infty]$, такая что $x_j \rightarrow x_0$ при $j \rightarrow \infty$. Утверждается, что $x_0 \in [0, s_\infty)$. Если $x_0 = s_\infty$, то $x_j - s_1(t_j) \rightarrow 0$ при $j \rightarrow \infty$. Используя (4.6) получим

$$\left| \frac{4\sigma}{x_j - s_1(t_j)} \right| \leq \left| \frac{u(t_j, x_j)}{x_j - s(t_j)} \right| = \left| \frac{u(t_j, x_j) - u(t_j, s(t_j))}{x_j - s(t_j)} \right| = |u_x(t_j, \bar{x}_j)| \leq M_5,$$

где $\bar{x}_j \in (x_j, s(t_j))$. Это противоречие, так как $x_j - s(t_j) \rightarrow 0$.

В силу (4.6), существует $\delta > 0$ такое что $x_0 + \delta < s_\infty$ и

$$u(t_j, x) \geq 2\sigma, \quad x \in [x_0, x_0 + \delta]$$

для всех больших j . Так как $s(t_j) \rightarrow s_\infty$ при $j \rightarrow +\infty$, отсюда следует, что $s(t_j) > x_0 + \delta$ для всех j . Пусть

$$r_j(t) = x_0 + \delta + t - t_j, \quad t_j^* = \inf\{t > t_j : s(t) = r_j(t)\}.$$

Очевидно, $x_0 + \delta + t_j^* - t_j = r_j(t_j^*) < s_\infty$. Введем

$$y_j(t, x) = (\pi - \theta) \frac{2(x - x_0) - (\theta + t - t_j)}{\theta + t - t_j},$$

$$\underline{u}_j(t, x) = \sigma e^{-K(t - t_j)} (\cos y_j(t, x) + \cos \theta), \quad (t, x) \in \bar{\Omega}_{t_j},$$

где θ ($0 < \theta < \frac{\pi}{8}$) и K — положительные константы, которые будут выбраны ниже, и

$$\Omega_{t_j} = \{(t, x): t_j < t < t_j^*, x_0 < x < r_j(t)\}.$$

Понятно, что $\underline{u}_j(t, x_0) = \underline{u}_j(t, r_j(t)) = 0$ и $|y_j(t, x)| \leq \pi - \theta$ для $(t, x) \in \bar{\Omega}_{t_j}$, что означает $\underline{u}_j(t, x) \geq 0$ в $\bar{\Omega}_{t_j}$.

Теперь мы сравним $u(t, x)$ и $\underline{u}_j(t, x)$ всюду в Ω_{t_j} . Согласно (2.1)–(2.3) следует, что

$$u(t, x_0) > 0 = \underline{u}_j(t, x_0), \quad u(t, r_j(t)) \leq 0 = \underline{u}_j(t, r_j(t)), \quad t \in [t_j, t_j^*].$$

С другой стороны

$$u(t_j, x) \geq 2\sigma > \underline{u}_j(t_j, x), \quad x \in [x_0, x_0 + \delta].$$

Таким образом, если положительные константы θ и K могут быть выбраны независимо от j , так что

$$\underline{u}_{jt} \leq d_1 \underline{u}_{jxx} + m_1 \underline{u}_j \underline{u}_{jx} + \underline{u}_j(a_1 - b_1 \underline{u}_j - c_1 M_2), \quad t > 0, \quad \Omega_{t_j},$$

выведем, что $\underline{u}_j(t, x) \leq u(t, x)$ для $(t, x) \in \Omega_{t_j}$ применяя принцип сравнения с u и $\bar{u}_j$. Поскольку $\underline{u}_j(t_j^*, s(t_j^*)) = 0 = u(t_j^*, r_j(t_j^*))$ и $s(t_j^*) = r_j(t_j^*)$, то $\underline{u}_{jx}(t_j^*, s(t_j^*)) \leq u_{jx}(t_j^*, r_j(t_j^*))$. Из-за $\theta < \frac{\pi}{8}$ и $x_0 + \delta + t_j^* - t_j < s_\infty$ имеем

$$\underline{u}_{jx}(t_j^*, r_j(t_j^*)) = -\frac{2\sigma(\pi - \theta)}{x_0 + \delta + t_j^* - t_j} e^{-K(t_j^* - t_j)} \sin(\pi - \theta) \leq -\frac{7\sigma\pi}{4s_\infty} e^{-Ks_\infty} \sin\theta.$$

Замечая, что $-\mu \underline{u}_{jx}(t_j^*, s(t_j^*)) \leq s'(t_j^*)$, получим

$$s'(t_j^*) \geq \frac{7\sigma\pi}{4s_\infty} e^{-Ks_\infty} \sin\theta,$$

что означает $\limsup_{t \rightarrow \infty} s'(t) > 0$ при $\lim_{j \rightarrow \infty} t_j^* = \infty$. Это противоречит $\limsup_{t \rightarrow \infty} s'(t) = 0$, поэтому

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0.$$

Теперь мы докажем, что если θ и K удовлетворяет

$$\sin\theta \leq \frac{3\delta^2 d_1 \pi}{4s_\infty^2(s_\infty + 2m_1 \sigma \delta \pi)}, \quad (4.8)$$

$$K > 2\sigma b_1 + c_1 M_2 + d_1 \left(\frac{2\pi}{\delta}\right)^2 + \frac{2\pi s_\infty + 4m_1 \sigma \delta \pi}{\delta^2(\cos\theta - \cos 2\theta)} \quad (4.9)$$

то (4.7) выполняется для всех больших j . Итак мы имеем $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0$. Поскольку $0 \leq \underline{u}_j \leq 2\sigma$ и $\delta + t_j^* - t_j < s_\infty$ получим

$$\begin{aligned} & \underline{u}_{jt} - d_1 \underline{u}_{jxx} - m_1 \underline{u}_j \underline{u}_{jx} - \underline{u}_j(a_1 - b_1 \underline{u}_j - c_1 M_2) \\ &= -K \underline{u}_j - \sigma e^{-K(t-t_j)} y_{jt} \sin y_j + d_1 \sigma e^{-K(t-t_j)} y_{jx}^2 \cos y_j + \underline{u}_j \sigma e^{-K(t-t_j)} y_{jx} \sin y_j - \underline{u}_j(a_1 - b_1 \underline{u}_j - c_1 M_2) \\ &\leq (2\sigma b_1 + c_1 M_2 + d_1 y_{jx}^2 - K) \underline{u}_j - \sigma e^{-K(t-t_j)} (d_1 y_{jx}^2 \cos\theta + y_{jt} \sin y_j - 2\sigma m_1 y_{jx} \sin y_j) \\ &\leq \left(2\sigma b_1 + c_1 M_2 + d_1 \left(\frac{2\pi}{\delta}\right)^2 - K\right) \underline{u}_j \\ &\quad + \sigma e^{-K(t-t_j)} \left[\frac{2\pi(x-x_0) + 4m_1 \sigma \delta (\pi - \theta)}{\delta^2} |\sin y_j| - d_1 \left(\frac{2(\pi - \theta)}{s_\infty}\right)^2 \cos\theta \right] \equiv I, \quad (t, x) \in \Omega_{t_j}. \end{aligned}$$

В силу (4.9), имеем $2\sigma b_1 + c_1 M_2 + d_1 \left(\frac{2\pi}{\delta}\right)^2 - K < 0$. Поскольку $-(\pi - \theta) \leq y_j \leq \pi - \theta$ при $(t, x) \in \Omega_{t_j}$, можем разложить $\Omega_{t_j} = \Omega_{t_j}^1 \cup \Omega_{t_j}^2$, где

$$\Omega_{t_j}^1 = \{(t, x) \in \Omega_{t_j}: t_j < t < t_j^*, \pi - 2\theta < |y_j(t, x)| < \pi - \theta\},$$

$$\Omega_{t_j}^2 = \{(t, x) \in \Omega_{t_j}: t_j < t < t_j^*, |y_j(t, x)| < \pi - 2\theta\}.$$

Очевидно, что $|\sin y_j| \leq \sin 2\theta$ при $(t, x) \in \Omega_{t_j}^1$, и $\cos y_j \geq -\cos 2\theta$ при $(t, x) \in \Omega_{t_j}^2$. Из $\theta < \frac{\pi}{8}$, $\underline{u}_j(t, x) \geq 0$ и $x - x_0 \leq s_\infty$ в Ω_{t_j} , с учетом (4.8), (4.9), заключаем, что

$$I \leq \sigma e^{-K(t-t_j)} \left(\frac{2\pi s_\infty + 4m_1 \sigma \delta \pi}{\delta^2} \sin 2\theta - \frac{3d_1 \pi^2}{s_\infty^2} \cos\theta \right) \leq 0$$

при $(t, x) \in \Omega_{t_j}^1$, и

$$I \leq \sigma e^{-K(t-t_j)} \left(\left[2\sigma b_1 + c_1 M_2 + d_1 \left(\frac{2\pi}{\delta} \right)^2 - K \right] (\cos\theta - \cos 2\theta) + \frac{2\pi s_{00} + 4m_1 \sigma \delta \pi}{\delta^2} \right) < 0$$

при $(t, x) \in \Omega_{t_j}^2$. □

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IRRATSIONAL FUNKSIYALAR BILAN BOG'LIQ EKSTREMAL MASALALAR VA TENGLAMALARNI YECHISHNING GEOMETRIK USULLARI

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Annotatsiya. Mazkur maqolada irratsional funksiyalar bilan bog'liq ekstremal masalalar hamda tenglamalarni yechishning geometrik usullari o'rganiladi. Ishda irratsional funksiyalarning asosiy xossalari, ularning grafiklari va bu funksiyalar yordamida ekstremal qiymatlarni aniqlash usullari tahlil qilinadi. Shuningdek, tenglamalarni yechishda geometrik yondashuvning ahamiyati, grafiklar yordamida yechimlarni aniqlash usullari ko'rib chiqiladi. Maqolada nazariy ma'lumotlar bilan bir qatorda amaliy misollar ham keltirilib, ularni yechish bosqichlari tushuntiriladi. Ushbu ish irratsional funksiyalarni chuqurroq o'rganish, matematik tafakkurni rivojlantirish hamda tenglama va ekstremal masalalarni samarali yechish ko'nikmalarini shakllantirishga xizmat qiladi.

Kalit so'zlar: Funksiya, irratsional, ekstremal masalalar, tenglama, geometrik usullar, algebraik masalalar, Pifagor teoremasi, uchburchaklar, metodika.

ЭКСТРЕМАЛЬНЫЕ ЗАДАЧИ, СВЯЗАННЫЕ С ИРРАЦИОНАЛЬНЫМИ ФУНКЦИЯМИ, И ГЕОМЕТРИЧЕСКИЕ МЕТОДЫ РЕШЕНИЯ УРАВНЕНИЙ

Аннотация. В данной статье рассматриваются экстремальные задачи, связанные с иррациональными функциями, а также геометрические методы решения уравнений. В работе анализируются основные свойства иррациональных функций, исследуются их графики и методы определения экстремальных значений с использованием данных функций. Особое внимание уделяется роли геометрического подхода при решении уравнений, а также способам нахождения их решений с помощью графического метода. Наряду с теоретическими положениями в статье приводятся практические примеры, подробно раскрываются этапы их решения. Результаты исследования способствуют более глубокому изучению иррациональных функций, развитию математического мышления, а также формированию эффективных навыков решения уравнений и экстремальных задач.

Ключевые слова: функция, иррациональное число, экстремальные задачи, уравнение, геометрические методы, алгебраические задачи, теорема Пифагора, треугольники, методология.

EXTREMAL PROBLEMS RELATED TO IRRATIONAL FUNCTIONS AND GEOMETRIC METHODS FOR SOLVING EQUATIONS

Abstract. This article examines extremum problems related to irrational functions and geometric methods for solving equations. The study analyzes the main properties of irrational functions, investigates their graphs, and discusses methods for determining extreme values using these functions. Particular attention is given to the role of the geometric approach in solving equations and identifying their solutions through graphical methods. In addition to theoretical concepts, the article provides practical examples and explains the stages of their solutions in detail. The results of the research contribute to a deeper understanding of irrational functions, the development of mathematical thinking, and the formation of effective skills for solving equations and extremum problems.

Keywords: Function, irrational, extremal problems, equation, geometric methods, algebraic problems, Pythagorean Theorem, triangles, methodology.

Ba'zi algebraik masalalarning geometrik yechimi, jumladan, irratsional funksiyalar bilan bog'liq ekstremal masalalarni yechishning geometrik usullari haqida ma'lumotlar beramiz. O'quvchilarning matematika faniga bo'lgan qiziqishlarini oshirishda, matematik tafakkurlarini o'stirishda, algebra bilan geometriya fanlarining o'zaro uzviy aloqadorliklarini yaqqol namoyon qilishda bu mavzu katta ahamiyat kasb etadi. Uni o'rganish, u haqida bilimga ega bo'lish, tassavur qila olish, uni mohiyat jihatidan tushunish va amalda qo'llay olish katta ahamiyatga ega va shu bilan birga, xususiyatlarini o'rganish va metodikasini

ishlab chiqish va uni berish usullarini ko'rsatib berish zaruriy talablardan biri hisoblanadi. Ayni shu ahamiyat va zarurat tayyorlangan bitiruv malakaviy ishini dolzarbligini belgilaydi.

1-misol. $f(x) = \sqrt{x^4 + 4} + \sqrt{x^2 - 3x\sqrt{3} + 9}$ funksiyaning eng kichik qiymatini toping.

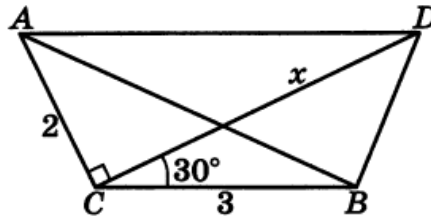
Yechish.

1-chizmada

$\triangle ACD$ ($AC = 2, CD = x, \angle ACD = 90^\circ$)

va

$\triangle BCD$ ($BC = 3, CD = x, \angle BCD = 30^\circ$)



1-chizma

$\triangle ACD$ dan Pifagor teoremasiga ko'ra $AD = \sqrt{x^2 + 4}$

$\triangle BCD$ dan kosinuslar teoremasiga ko'ra $DB = \sqrt{x^2 + 9 - 3x\sqrt{3}}$

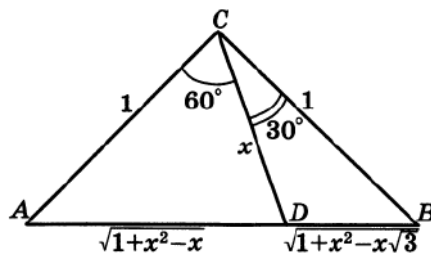
$$\min f(x) = \min(AD + DB) = AB$$

$\triangle ABC$ uchburchakdan kosinuslar teoremasiga ko'ra

$$AB = \sqrt{2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cos 120^\circ} = \sqrt{19}.$$

2-misol. x ning qanday qiymatlarida $f(x) = \sqrt{1+x^2-x} + \sqrt{1+x^2-x\sqrt{3}}$ funksiya o'zining eng kichik qiymatiga erishadi?

Yechish. 2-chizmadan $\triangle ABC$, $\triangle ACD$ va $\triangle BCD$ uchburchaklar yuzini topamiz.



2-chizma

$$S_{\triangle ABC} = \frac{1}{2}.$$

$$S_{\triangle ACD} = \frac{1}{2} \cdot 1 \cdot x \cdot \sin 60^\circ = \frac{x\sqrt{3}}{4}.$$

$$S_{\triangle BCD} = \frac{1}{2} \cdot 1 \cdot x \cdot \sin 30^\circ = \frac{x}{4}.$$

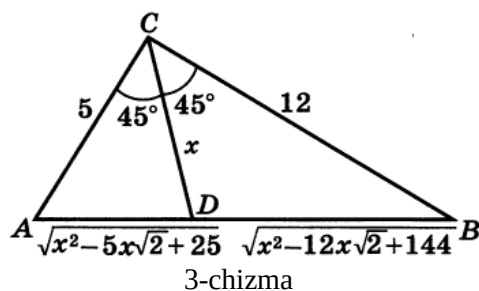
$$\text{Demak, } \frac{1}{2} = \frac{x\sqrt{3}}{4} + \frac{x}{4}; \quad x = \sqrt{3} - 1.$$

Javob: $\sqrt{3} - 1$.

3-misol. $\sqrt{x^2 - 5x\sqrt{2} + 25} + \sqrt{x^2 - 12x\sqrt{2} + 144} = 13$ tenglamani yeching.

Yechish. Bu tenglamaning an'anaviy usulda yechish uchun har tomoni ildizlar yo'qolmaguncha kvadratga ko'tariladi. Tenglamani geometrik usulda yechish qisqa muddatda amalga oshiriladi. 3-chizmada $\triangle ABC$ dan Pifagor teoremasiga ko'ra $AB = 13$.

$D \in AB$, chunki $\min f(x) = \min(AD + DB) = AB$.



3-chizma

Bunda $f(x) = \sqrt{x^2 - 5x\sqrt{2} + 25} + \sqrt{x^2 - 12x\sqrt{2} + 144}$ x ning qiymatini topish uchun 2-misoldagidek ish yuritimiz. Agar tenglama ildizlarga ega bo'lsa, ular musbat bo'lishni aniqlaymiz. ($x \leq 0$ da tenglama chap qismining qiymati 12 dan kichik emas).

$$S_{\triangle ABC} = \frac{1}{2} \cdot 5 \cdot 12 = 30.$$

$$S_{\triangle ACD} = \frac{1}{2} \cdot 5 \cdot x \cdot \sin 45^\circ = \frac{5x\sqrt{2}}{4}.$$

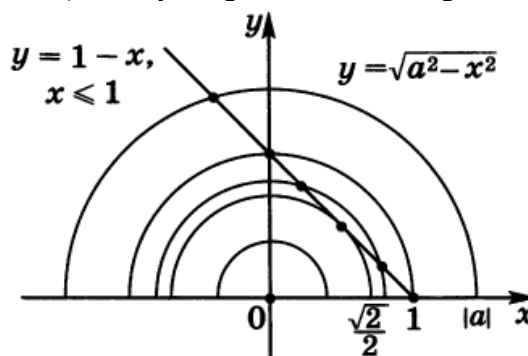
$$S_{\triangle BCD} = \frac{1}{2} \cdot 12 \cdot x \cdot \sin 45^\circ = 3x\sqrt{2}.$$

$$\text{Demak, } 30 = \frac{5x\sqrt{2}}{4} + 3x\sqrt{2}; \quad x = \frac{60\sqrt{2}}{17}.$$

$$\text{Javob: } \frac{60\sqrt{2}}{17}.$$

4-misol. a parametrning har bir qiymatida $\sqrt{a^2 - \operatorname{tg}^2 z} = 1 - \operatorname{tg} z$ tenglamada $\operatorname{tg} z$ nechta har xil qiymatlar qabul qiladi.

Yechish. Bu oddiy tenglama emas. Uni an'anaviy usulda yechish oson emas, agar $\operatorname{tg} z$ ni x bilan almashtirsak, uni geometrik usulda yechish oson bo'ladi. 4-chizmada $y = \sqrt{a^2 - x^2}$ (markazi koordinatalar boshida, radiusi $|a|$ bo'lgan yuqori yarim tekislikda joylashgan yarim aylana) va $y = 1 - x$ ($x \leq 1$) (boshi (1;0) nuqtada (0,1) nuqtadan o'tuvchi nur) funksiyalar grafiklari tasvirlangan.



4-chizma

$$\text{Demak, } \sqrt{a^2 - x^2} = 1 - x$$

$$|a| < \frac{\sqrt{2}}{2} \text{ da ildiz yo'q.}$$

$$|a| = \frac{\sqrt{2}}{2} \text{ va } |a| > 1 \text{ da 1 ta ildizga ega.}$$

$$\frac{\sqrt{2}}{2} < |a| \leq 1 \text{ da 2 ta ildizga ega.}$$

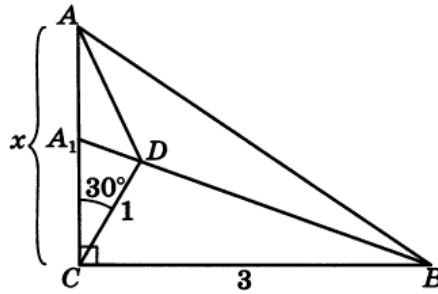
Demak, $tgz : |a| = \frac{\sqrt{2}}{2}$ da bitta qiymat qabul qiladi. ($-\frac{\sqrt{2}}{2}$ da bitta son bo'lsa, $\frac{\sqrt{2}}{2}$ da boshqa

sondir), $|a| > 1$ da bitta qiymat qabul qiladi, $\frac{\sqrt{2}}{2} < |a| \leq 1$ da 2 ta har xil qiymat qabul qiladi.

5-misol. $f(x) = \sqrt{x^2 + 9} - \sqrt{x^2 - x\sqrt{3} + 1}$ funksiyaning eng katta qiymatini toping.

Yechish. ABC uchburchakni qaraymiz. Bunda

$\angle ACB = 90^\circ$, $\angle ACD = 30^\circ$, $AC = x$, $BC = 3$, $CD = 1$ va D nuqta ABC uchburchak ichida yotadi. (5-chizma)



5-chizma

ABC uchburchakdan Pifagor teoremasiga ko'ra $AB = \sqrt{x^2 + 9}$.

$\triangle ACD$ dan kosinuslar teoremasiga ko'ra

$$AD = \sqrt{x^2 - x\sqrt{3} + 1} \quad \max f(x) = \max(AB - AD) = A_1B - A_1D = DB,$$

bunda $A_1 \in AC$ (agar $D \in AB$).

$\triangle BCD$ dan kosinuslar teoremasiga ko'ra

$$DB = \sqrt{1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cos 60^\circ} = \sqrt{7}.$$

Javob: $\sqrt{7}$.

6-misol. $f(x) = x\sqrt{2} + \sqrt{2x^2 + 2x + 1} + \sqrt{2x^2 - 14x + 25} + \sqrt{2x^2 - 26x + 89}$ funksiyaning eng kichik qiymatini toping.

Yechish. $\vec{a}(x; x)$, $\vec{b}(4 - x; 3 - x)$, $\vec{c}(x + 1; x)$, $\vec{d}(5 - x; 8 - x)$ vektorlar va ularning modullarini qaraymiz.

$$|\vec{a}| = x\sqrt{2}.$$

$$|\vec{b}| = \sqrt{2x^2 - 14x + 25}.$$

$$|\vec{c}| = \sqrt{2x^2 + 2x + 1}.$$

$$|\vec{d}| = \sqrt{2x^2 - 26x + 89}.$$

$$f(x) = |\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{d}|.$$

$$|\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{d}| \geq |\vec{a} + \vec{b} + \vec{c} + \vec{d}| \quad \text{bo'lgani uchun}$$

$$\min f(x) = |\vec{a} + \vec{b} + \vec{c} + \vec{d}|.$$

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} = (10; 11)$$

$$|\vec{a} + \vec{b} + \vec{c} + \vec{d}| = \sqrt{221}.$$

Demak, $\min f(x) = \sqrt{221}$.

Javob: $\sqrt{221}$.

7-misol. $f(x) = \sqrt{2x^2 - 4x + 2} + \sqrt{2x^2} + \sqrt{2x^2 - 2x + 2}$ funksiyaning eng kichik qiymatini toping.

Yechish. 4 ta $O(0;0)$, $A(1;1)$, $B\left(\frac{1-\sqrt{3}}{2}; \frac{1+\sqrt{3}}{2}\right)$, $M(x;x)$ va OM , AM va BM masofalarni

qaraymiz. $OM = \sqrt{2x^2}$, $AM = \sqrt{2x^2 - 4x + 2}$, $BM = \sqrt{2x^2 - 2x + 2}$.

$$f(x) = OM + AM + BM.$$

ABO uchburchakning tomonlarini hisoblash qiyin emas.

$AO = BO = AB = \sqrt{2}$. Demak, $\triangle ABO$ teng tomonli

$$\min(AM + OM + BM) = 3R,$$

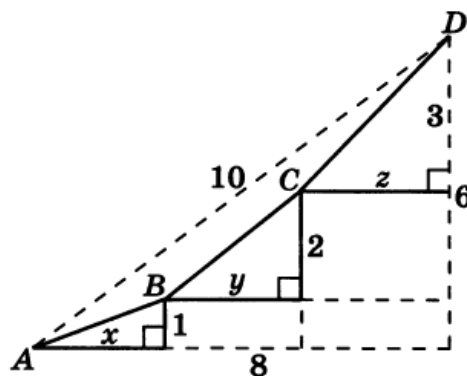
R-teng tomonli uchburchakka tashqi chizilgan aylana radiusi.

$\sqrt{2} = R\sqrt{3}$ bo'lgani uchun $\min f(x) = \sqrt{6}$.

Javob: $\sqrt{6}$.

8-misol. $f(x; y; z) = \sqrt{x^2 + 1} + \sqrt{y^2 + 4} + \sqrt{z^2 + 9}$ va $x + y + z = 8$ bo'lsa, $f(x; y; z)$ funksiyaning kichik qiymatini toping.

Yechish.



6-chizma

6-chizmaga ko'ra $ABCD$ egri chiziq uzunligi 10 dan kichik emas, u holda $x + y + z = 8$ da $f(x; y; z) = 10$.

Javob: 10.

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KVADRATUR FORMULA KOEFFITSIYENTLARINI ANIQLASH

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Annotatsiya. Ushbu tadqiqotda umumlashtirilgan Abel integral tenglamalarining taxminiy yechimlarini olish uchun optimal kvadratura formulalaridan foydalanishga asoslangan yangi va samarali yondashuv taklif etiladi. Ushbu usul yuqori aniqlik va soddalik bilan ajralib turadi hamda Sard ma'nosidagi optimal kvadratura formulalarini qurishni o'z ichiga oladi. Shuningdek, differensiallanuvchi funksiyalar Hilbert fazosida xatolik baholari aniqlanadi. $W_2^{(2,1)}(0, t)$ fazosida kvadratura formulasining xatolik funksionali kvadrat normasi hisoblab chiqiladi. Ushbu xatolikni minimallashtirish uchun formula ko'effitsiyentlariga nisbatan chiziqli tenglamalar tizimi tuziladi va natijada yagona yechim olinadi. So'ng optimal ko'effitsiyentlarning aniq ifodalari aniqlanadi. Taklif etilgan usulning samaradorligi bir nechta integral tenglamalarni yechish orqali ko'rsatiladi va yaqinlashish xatolari tegishli jadvallarda keltiriladi.

Kalit so'zlar: Hilbert fazosi, Abel integral tenglamasi, xatolik funksionali, optimal kvadratura formulasi, xatolik funksionali normasi.

ОПРЕДЕЛЕНИЕ КОЭФФИЦИЕНТОВ КВАДРАТУРНОЙ ФОРМУЛЫ

Аннотация. В данной работе предлагается новый и эффективный подход, основанный на использовании оптимальных квадратурных формул для получения приближённых решений обобщённых интегральных уравнений Абеля. Предложенный метод характеризуется высокой точностью и простотой и включает построение оптимальных квадратурных формул в смысле Сарда. Также приводятся оценки погрешности в гильбертовом пространстве дифференцируемых функций. Вычисляется квадрат нормы функционала ошибки квадратурной формулы в пространстве $W_2^{(2,1)}(0, t)$. Для минимизации этой ошибки выводится система линейных уравнений относительно коэффициентов формулы, что приводит к единственному решению. Далее получают явные выражения для оптимальных коэффициентов. Эффективность предложенного метода демонстрируется на решении нескольких интегральных уравнений, а ошибки аппроксимации приводятся в соответствующих таблицах.

Ключевые слова: гильбертово пространство, интегральное уравнение Абеля, функционал ошибки, оптимальная квадратурная формула, норма функционала ошибки.

DETERMINATION OF THE COEFFICIENTS OF A QUADRATURE FORMULA

Abstract. In this study, a novel and efficient approach utilizing optimal quadrature formulas is introduced to derive approximate solutions for generalized Abel integral equations. The method is characterized by high accuracy and simplicity and involves constructing optimal quadrature formulas in the sense of Sard. Error estimates are provided in the Hilbert space of differentiable functions. The squared norm of the error functional of the quadrature formula in the space is computed. To minimize this error, a system of linear equations with respect to the coefficients of the formula is derived, leading to a unique solution. Explicit expressions for these optimal coefficients are then obtained. The effectiveness of the proposed approach is demonstrated by solving several integral equations, and the approximation errors are presented in the corresponding tables.

Keywords: Hilbert space, Abel integral equation, error functional, optimal quadrature formula, norm of the error functional.

Kirish. Abel tomonidan olingan integral tenglama (qarang [1]) uning nomi bilan ataladi va quyidagi ko'rinishga ega:

$$\int_0^x \frac{y(t)}{\sqrt{(x-t)}} dt = f(x), \quad 0 \leq t \leq x < \infty \quad (1)$$

bu yerda o'ng tomondagi $f(x)$ funksiya ma'lum, va (1) tenglama $y(t)$ ga nisbatan yechilishi kerak.

Shuni ta'kidlash muhimki, Abel Nyuton qonunlaridan foydalanish o'rniga tushayotgan jismlarning kinetik va potensial energiyalari o'rtasidagi bog'lanishdan foydalangan.

U zarrachaning doimiy vertikal tezlanish ostida tushishi uchun qanday trayektoriya bo'ylab harakatlanishi kerakligini aniqlash uchun shu usuldan foydalangan. Bu shunday bajarilganki, tushish vaqti tushilgan masofaning oldindan berilgan $f(x)$ funksiyasi bo'lsin.

Asosiy qism. Abel integral tenglamasi turli mustaqil boshlang'ich taxminlardan kelib chiqadigan matematik modellarni tuzishda tabiiy ravishda paydo bo'ladi.

Bunday tenglamalar quyidagi ko'plab sohalarni tahlil qilishda muhim ahamiyatga ega: geometrik ehtimollikdagi birinchi tur integral tenglamalar, interferometriya, stereologiya, seysmologiya, tomografiya

Masalan:

- Wicksell va Moran ishlari – geometrik ehtimollik
- Merzkirch – interferometriya
- Jeffreys va Bateman – seysmologiya
- Gordon – tomografiya

Haqiqatan ham, Abel integral tenglamalari proyeksiyalardan tiklash masalalarini o'rganish uchun tabiiy matematik asos yaratadi.

Quyidagi ko'rinishdagi tenglamalar

$$\int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x) \quad 0 < \alpha < 1$$

umumlashtirilgan Abel integral tenglamasi deb ataladi.

Quyidagi formula umumlashtirilgan Abel integral tenglamasining yechimini beradi:

$$y(t) = \frac{\sin(\alpha\pi)}{\pi} \left[\frac{f(0)}{t^{1-\alpha}} + \int_0^t \frac{f'(z)}{(t-z)^{1-\alpha}} dz \right] \quad (2)$$

Bu formula yanada umumiy shartlar ostida ham Abel integral tenglamasining yechimini beradi.

(2) tenglamaning o'ng tomonidagi integralni hisoblash orqali Abel integral tenglamasining yechimini soddaroq ko'rinishda topish mumkin.

Ushbu ishning maqsadi quyidagi integralni yuqori aniqlik bilan hisoblashdir:

$$\int_0^t \frac{f'(z)}{(t-z)^{1-\alpha}} dz, \quad 0 < \alpha < 1$$

Bu integral Riemann–Liouville kasr tartibli integralining muhim **qismi** hisoblanadi.

Endi Riemann–Liouville kasr tartibli integrali tushunchasini ko'rib chiqamiz.

$f(t)$ funksiyaning tartibi $\alpha > 0$ bo'lgan chap Riemann–Liouville integrali quyidagicha aniqlanadi.

$$D_{0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx$$

Bu yerda $\Gamma(\alpha)$ Eylerning gamma funksiyasi hisoblanadi.

Amaliy masalalarda kasr tartibli differensial yoki integral tenglamalarning aniq analitik yechimlarini olish ko'pincha qiyin yoki imkonsiz bo'ladi.

Ba'zan analitik yechim mavjud bo'lsa ham, u amaliy hisoblashlar uchun juda murakkab bo'lishi mumkin.

Shuning uchun aniq va samarali sonli usullarga ehtiyoj katta.

Shuningdek muhim masalalardan biri — Riemann–Liouville integralini yuqori aniqlik bilan sonli integrallash hisoblanadi.

Bu yo'nalishda ko'plab tadqiqotlar olib borilgan va turli kvadratur formulalar ishlab chiqilgan.

1. Masalaning qo'yilishi

Biz quyidagi ko'rinishdagi **kvadratur formulani** qaraymiz

$$\int_0^t \frac{\varphi(x) dx}{(t-x)^{1-\alpha}} \cong \sum_{\beta=0}^N c_\beta \varphi(x_\beta) \quad (3)$$

bu yerda $0 < \alpha < 1$

c_β va x_β — mos ravishda **kvadratur formula (3) ning koeffitsiyentlari va tugunlari**,

$N+1$ esa kvadratur tugunlarining umumiy sonidir.

Faraz qilaylik $\varphi(x)$ funksiya $W_2^{(2,1)}(0,t)$ fazosiga tegishli bo'lsin. Bu fazo quyidagicha aniqlanadi:

$$W_2^{(2,1)}(0, t) = \{\varphi \mid \varphi: [0, t] \rightarrow \mathbb{R}, \varphi' \text{ absolyut funksiya va } \varphi'' \in L_2(0, t)\}$$

$W_2^{(2,1)}(0, t)$ fazosida $\varphi(x)$ va $\psi(x)$ funksiyalarning **skalyar ko'paytmasi** quyidagicha aniqlanadi:

$$(\varphi, \psi) = \int_0^t (\varphi''(x) + \varphi'(x)(\psi''(x) + \psi'(x))) dx \quad (4)$$

Ushbu skalyar ko'paytma mos keluvchi **norma** quyidagicha aniqlanadi

$$\|\varphi\|_{W_2^{(2,1)}} = \sqrt{\int_0^t (\varphi''(x) + \varphi'(x))^2 dx}$$

Integral bilan kvadratur yig'indisi orasidagi farq kvadratur formula (3) ning **xatoligi** deb ataladi:

$$(l, \varphi) = \int_0^t \frac{\varphi(x)}{(t-x)^{1-\alpha}} dx - \sum_{\beta=0}^N c_\beta \varphi(x_\beta) \quad (5)$$

Bu farq quyidagi ko'rinishdagi **xatolik funksionaliga** mos keladi

$$l(x) = \frac{\varepsilon_{[0,t]}(x)}{(t-x)^{1-\alpha}} - \sum_{\beta=0}^N c_\beta \delta(x - x_\beta) \quad (6)$$

bu yerda

$\varepsilon_{[0,t]}(x)$ - $[0, t]$ oraliqning **xarakteristik funksiyasi**,

$\delta(x)$ — **Dirak delta funksiyasi**.

Koshi–Shvarts tengsizligiga ko'ra quyidagi munosabat o'rinli

$$|(l, \varphi)| \leq \|l\|_{W_2^{(2,1)*}} \|\varphi\|_{W_2^{(2,1)}}$$

bu esa (5) dagi xatolikning mutlaq qiymatini xatolik funksionalining normasi orqali baholash mumkinligini ko'rsatadi.

Bu yerda

$$\|l\|_{W_2^{(2,1)*}} = \max_{\varphi, \|\varphi\| \neq 0} \frac{|(l, \varphi)|}{\|\varphi\|}$$

Shunday qilib $W_2^{(2,1)}$ fazosida kvadratur formula (3) ning **xatoligini baholash masalasi** xatolik

funksionali l ning $W_2^{(2,1)*}(0, t)$ **dual fazodagi normasini topish** masalasiga keltiriladi.

Yuqoridagi Koshi–Shvarts tengsizligidan kelib chiqadiki, xatolik funksionali l ning $W_2^{(2,1)}$ fazosida aniqlangan bo'lib, u quyidagi shartlarni qanoatlantiradi:

$$(l, e^{-x}) = 0 \quad (7)$$

$$(l, 1) = 0 \quad (8)$$

(7) va (8) tengliklar shuni ko'rsatadiki, kvadratur formula (3)

$$de^{-x} + P_0$$

ko'rinishdagi istalgan chiziqli kombinatsiya uchun **aniq hisoblanadi**.

Bu shartlar **xatolik funksionalining chegaralangan bo'lishini ta'minlaydi**.

Kvadratur formulalar aniq integrallarni sonli hisoblashda qo'llaniladi hamda differensial va integral tenglamalarni sonli yechishda muhim ahamiyatga ega.

Hisoblash matematikasining muhim tadqiqot yo'nalishlaridan biri — **optimal kvadratur formulalarni qurish algoritmlarini ishlab chiqish** hisoblanadi.

Bu esa turli funksiyalar sinfi uchun xatolikni baholashni **algebraik va variatsion usullar yordamida** o'rganishni talab qiladi.

Numerik integrallash formulalarini optimallashtirishning variatsion yondashuvda asosiy maqsad — berilgan funksiyalar fazosida **xatolik funksionalining normasini minimallashtirish** hisoblanadi.

Nikolskiy masalasi [29,30][29,30][29,30] kvadratur formuladagi **ham koeffitsiyentlar** C_β ham **tugunlar** x_β ni o'zgartirish orqali xatolik normasini kamaytirishga qaratilgan.

Sard masalasida esa [32,33,34][32,33,34][32,33,34] faqat **koeffitsiyentlar** C_β o'zgartiriladi, tugunlar esa **o'zgarmas holda qoldiriladi**.

Nikolskiy va Sard masalalarining yechimlari mos ravishda

- **Nikolskiy ma'nosidagi optimal kvadratur formulalar**

- **Sard ma'nosidagi optimal kvadratur formulalar**

deb ataladi.

Sard ma'nosidagi optimal kvadratur formulalarni qurishning bir necha usullari mavjud.

$W_2^{(m,m-1)}(a,b)$ fazosida bu masala ko'plab mualliflar tomonidan o'rganilgan.

Bu yerda $W_2^{(m,m-1)}(a,b)$ — m-tartibli umumlashgan hosilasi kvadrat integrallanuvchi bo'lgan

funksiyalarning **Hilbert fazosi**.

Mazkur ishda biz $W_2^{(2,1)}(0,t)$ fazoda **Sard ma'nosidagi optimal kvadratur formulani** quramiz.

Bunda **Sobolev usulidan foydalaniladi** va

$$x_\beta = h\beta$$

teng masofali tugunlar olinadi, bu yerda $h = \frac{1}{N}$

Biz quyidagi shartni qanoatlantiruvchi C_β koeffitsiyentlarni topamiz:

$$\|l\|_{W_2^{(2,1)*}} = \inf_{C_\beta} \|l\|_{W_2^{(2,1)*}} \quad (9)$$

Shunday qilib, $W_2^{(2,1)}(0,t)$ fazoda Sard ma'nosidagi optimal kvadratur formulani qurish uchun

quyidagi masalalarni yechish zarur.

1-masala. Kvadratur formula (3) uchun **xatolik funksionalining normasini aniqlash**.

2-masala (9) tenglik bajariladiga C_β **koeffitsiyentlarni aniqlash**.

Kvadratur formula xatolik funksionalining normasi

Biz (3) kvadratur formulasi uchun **xatolik funksionalining normasi ifodasini** keltiramiz. Xatolik funksionalining normasi ifodasini topish masalasi [7] ishda bayon qilingan.

1-masalaning yechimi quyidagicha.

Teorema 1 $W_2^{(2,1)}(0,t)$ fazosida xatolik funksionali l ning **kvadrati normasi** quyidagi ko'rinishga ega:

$$\begin{aligned} \|l\|_{W_2^{(2,1)*}}^2 = & \sum_{\beta=0}^N \sum_{\gamma=0}^N c_\beta c_\gamma G_2(h_\beta - h_\gamma) - 2 \sum_{\beta=0}^N c_\beta \int_0^t \frac{G_2(x - h_\beta)}{(t-x)^{1-\alpha}} dx + \\ & + \int_0^t \int_0^t \frac{G_2(x-y)}{(t-x)^{1-\alpha}(t-y)^{1-\alpha}} dy dx \end{aligned} \quad (10)$$

bu yerda

$$G_2(x) = \frac{\operatorname{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} - x \right) \quad (11)$$

bo'ladi.

Kvadratur formula uchun optimal koeffitsiyentlar. Keyingi bosqichda berilgan fazoda (3) kvadratur formula uchun **2-masalani** yechamiz. Xatolik funksionali l ning (7) va (8) shartlarni qanoatlantiradi. Xatolik funksionalining kvadrati (10) quyidagi ko'rinishga ega bo'lib, u **ko'p o'zgaruvchili funksiya** hisoblanadi va kvadratur formula koeffitsiyentlari C_β bog'liq ($\beta = 0, 1, 2, \dots, N$). (10) ifodaning shartli minimumini topish uchun (7) va (8) shartlarni hisobga olib **Lagrange noma'lum ko'paytuvchilar usulidan** foydalanamiz. Quyidagicha belgilaymiz:

$$C = (C_0, C_1, \dots, C_N)$$

Quyidagi funksiyani qaraymiz:

$$\Lambda(C, d, P_0) = \|l\|_{W_2^{(2,1)}}^2(l, e^{-x}) - 2P_0(l, 1)$$

bu yerda (5), (7) va (8) tengliklarni hisobga olib:

$$(l, 1) = \int_0^t \frac{1}{(t-x)^{1-\alpha}} dx - \sum_{\beta=0}^N c_\beta = 0$$

$$(l, e^{-x}) = \int_0^t \frac{e^{-x}}{(t-x)^{1-\alpha}} dx - \sum_{\beta=0}^N c_\beta e^{-h\beta} = 0$$

Shunda

$$\Lambda(C, d, P_0) = \|l\|_{W_2^{(2,1)}}^2(l, e^{-x}) - 2d \left[\int_0^t \frac{e^{-x}}{(t-x)^{1-\alpha}} dx - \sum_{\beta=0}^N c_\beta e^{-h\beta} \right] - 2P_0 \left[\int_0^t \frac{1}{(t-x)^{1-\alpha}} dx - \sum_{\beta=0}^N c_\beta \right]$$

Endi $\Lambda(C, d, P_0)$ funksiyaning **qisman hosilalarini** C_β, d va P_0 bo'yicha nolga tenglashtiramiz.

Natijada quyidagi **chiziqli tenglamalar sistemasini** olamiz:

$$\sum_{\gamma=0}^N C_\gamma G_2(h\beta - h\gamma) + d e^{-h\beta} + P_0 = \int_0^t \frac{G_2(x - h\beta)}{(t-x)^{1-\alpha}} dx, \quad \beta = \overline{0, N} \quad (12)$$

$$\sum_{\gamma=0}^N C_\gamma e^{-h\gamma} = \int_0^t \frac{e^{-x}}{(t-x)^{1-\alpha}} dx \quad (13)$$

$$\sum_{\gamma=0}^N C_\gamma = \int_0^t \frac{1}{(t-x)^{1-\alpha}} dx \quad (14)$$

bu yerda $hN = T$ va G_2 funksiya (11) tenglik orqali aniqlangan. Quyidagi belgilashlarni kiritamiz

(12), (13) va (14) tenglamalarning o'ng tomonlari uchun

$$\begin{aligned} f_2(h\beta) &= \int_0^t \frac{G_2(x - h\beta)}{(t-x)^{1-\alpha}} dx \\ &= \sum_{k=0}^N \frac{(-1)^k (et - h\beta - e^{h\beta-t})}{4k! (k + \alpha)} (2(t - h\beta)^{k+\alpha} - t^{k+\alpha}) \\ &\quad - \frac{2(t-h\beta)^\alpha (h\alpha\beta+t) - t^{1+\alpha}}{2\alpha(\alpha+1)} + \frac{2h\beta(t-h\beta)^\alpha - h\beta t^\alpha}{2\alpha} \quad \beta = \overline{0, N} \\ g &= \int_0^t \frac{e^{-x}}{(t-x)^{1-\alpha}} dx = \sum_{k=0}^{\infty} \frac{e^{-t} t^{k+\alpha}}{4k! (k + \alpha)} \\ g_0 &= \int_0^t \frac{1}{(t-x)^{1-\alpha}} dx = \frac{t^\alpha}{\alpha} \end{aligned}$$

[40] va [41] ishlarda shunga o'xshash tenglamalar sistemasining **yechim mavjudligi va yagona ekanligi** ko'rsatilgan. Natijada quyidagilar aniqlangan:

- yechim **xatolik funksionalining normasini minimallashtiradi**

- (7) va (8) shartlar bajariladi.

Bu esa shuni anglatadiki: (10) formuladagi **xatolik funksionalining normasi kvadrati** $\|l\|^2$

kvadratur formula koeffitsiyentlari $C_\beta(t)$ ga ($\beta = \overline{0, N}$) nisbatan **kvadratik funksiya** bo'lib, u **yagona**

minimumga ega. Bu minimum $C_\beta = \dot{C}_\beta$ qiymatda hosil bo'ladi. Shunday qilib sistemadagi noma'lumlar:

$C_\gamma(t)$, ($\gamma = \overline{0, N}$) hamda d, P_0 hisoblanadi. Agar (12)-(14) sistemadan $C_\gamma(t)$ topilsa, u holda $W_2^{(2,1)}(0, t)$

fazoda **optimal kvadratur formula** hosil qilinadi.

Sobolev usuli. Endi (12)-(14) sistemaning **analitik yechimini topamiz**. Buning uchun **S.L. Sobolev tomonidan kiritilgan diskret argument funksiyalari nazariyasidan** foydalanamiz.

Kvadratur formula tugunlari **teng oraliqda joylashgan hol** qaraladi: $h\beta = \beta h$, $h = \frac{t}{N}$

bu yerda N — natural son.

Diskret differensial operator. Quyidagi differensial operatorning diskret analogi kerak bo'ladi

$$\frac{d^4}{dx^4} - \frac{d^2}{dx^2}$$

Bu operatorning diskret analogi [39] ish natijalaridan $m=2$ uchun olinadi.

Teorema 2. Quyidagi tenglikni qanoatlantiruvchi

$$D_2(h\beta) * G_2(h\beta) = \delta_d(h\beta)$$

diskret operator quyidagi ko'rinishga ega

$$D_2(h\beta) = \frac{1}{P_2^{(2)}} \begin{cases} A_1 \lambda_1^{|\beta|-1}, & |\beta| \geq 2 \\ -2e^h + A_1, & |\beta| = 1 \\ 2C + \frac{A_1}{\lambda_1}, & \beta = 0 \end{cases}$$

bu yerda

$$P_2^{(2)} = 1 - e^{2h} + 2he^h$$

$$C = 1 + 2e^h + e^{2h} - \frac{e^h(\lambda_1^2 + 1)}{\lambda_1}$$

$$A_1 = \frac{2(\lambda_1 - 1)[\lambda_1(e^{2h} + 1) - e^h(\lambda_1^2 + 1)]}{\lambda_1 + 1}$$

$$\lambda_1 = \frac{h((e^{2h} + 1) - e^{2h} + 1 - (e^h - 1)\sqrt{h^2(e^h + 1)^2 + 2h(1 - e^{2h})})}{1 - e^{2h} + 2he^h}$$

Teorema 3. Diskret operator $D_2(h\beta)$ quyidagi xossalarga ega:

1. $D_2(h\beta) * e^{h\beta} = 0$
2. $D_2(h\beta) * e^{-h\beta} = 0$
3. $D_2(h\beta) * 1 = 0$
4. $D_2(h\beta) * G_2(h\beta) = \delta_d(h\beta)$

Endi (12) tenglamaning chap tomonini quyidagicha belgilaymiz:

$$U_2(h\beta) = \sum_{\gamma=0}^N C_\gamma G_2(h\beta - h\gamma) + de^{-h\beta} + P_0 \quad \beta = \overline{0, N} \quad (16)$$

$$U_2(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}g + e^{-h\beta}d^- + \frac{h\beta}{2}g_0 + p^-, & \beta < 0 \\ f_2(h\beta), & 0 \leq \beta \leq N \\ \frac{e^{h\beta}}{4}g + e^{-h\beta}d^+ - \frac{h\beta}{2}g_0 + p^+, & \beta > N \end{cases} \quad (17)$$

$$U_2(h\beta) = \begin{cases} -\frac{e^{h\beta}}{4}g + \frac{h\beta}{2}g_0 + f_2(0) + \frac{1}{4}g + d^-(e^{-h\beta} - 1), & \beta < 0 \\ f_2(h\beta), & 0 \leq \beta \leq N \\ \frac{e^{h\beta}}{4}g - \frac{h\beta}{2}g_0 + f_2(t) - \frac{e^t}{4}g + \frac{t}{2}g_0 + d^+(e^{-h\beta} - e^{-t}), & \beta > N \end{cases} \quad (18)$$

Natijada optimal koeffitsiyentlar uchun

$$C_\beta = D_2(h\beta) * U_2(h\beta)$$

formula hosil bo'ladi.

Demak, agar $U_2(h\beta)$ funksiyasi topilsa, optimal koeffitsiyentlar ham topiladi.

Teorema 4 (Optimal koeffitsiyentlar). $W_2^{(2,1)}(0, t)$ fazoda kvadratur formula (3) ning optimal koeffitsiyentlari quyidagicha aniqlanadi:

$$C_0 = \frac{1}{P_2^{(2)}} \left(\frac{A_1}{\lambda_1} (M_1 + \lambda_1^N N_1 \sum_{\gamma=0}^N \lambda_1^\gamma f_2(h\gamma)) + 2C f_2(0) - 2e^h (U_2(-h) + f_2(h)) \right)$$

$$C_\beta = \frac{1}{P_2^{(2)}} \left(\frac{A_1}{\lambda_1} \left(\lambda_1^\beta M_1 + \lambda_1^{N-\beta} N_1 + \sum_{\gamma=0}^N \lambda_1^{|\beta-\gamma|} f_2(h\gamma) \right) + 2C f_2(h\beta) \right. \\ \left. - 2e^h (f_2(h(\beta-1)) + (f_2(h(\beta+1)))) \right) \quad \beta = \overline{1, N-1}$$

$$C_N = \frac{1}{P_2^{(2)}} \left(\frac{A_1}{\lambda_1} (\lambda_1^N M_1 + N_1 + \sum_{\gamma=0}^N \lambda_1^{N-\gamma} f_2(h\gamma)) + 2C f_2(t) - 2e^h (f_2(t-h) + U_2(t+h)) \right)$$

bu yerda

$$M_1 = \lambda_1 \left[-\frac{ge^{-h}}{4(1-\lambda_1 e^{-h})} - \frac{hg_0}{2(1-\lambda_1)^2} + \frac{d^- e^h}{1-\lambda_1 e^h} + \frac{f_2(0) + \frac{g}{4} - d^-}{1-\lambda_1} \right]$$

$$N_1 = \lambda_1 \left[\frac{ge^{h+t}}{4(1-\lambda_1 e^h)} - \frac{hg_0}{2(1-\lambda_1)^2} + \frac{d^+ e^{-h-t}}{1-\lambda_1 e^{-h}} + \frac{f_2(t) - \frac{e^t g}{4} - d^+ e^{-t}}{1-\lambda_1} \right]$$

Shunday qilib **2-masala to'liq yechildi** va kvadratur formula (3) uchun **optimal koeffitsiyentlar aniqlandi**.

Ushbu ishda $N=1$ bo'lgan holda (12)-(14) sistemani qaraymiz va yechamiz. (13)-(14) tenglamalar sistemasini yechish kifoya.

$$\begin{cases} \sum_{j=0}^1 C_j e^{-h\gamma} = \int_0^t \frac{e^{-x}}{(t-x)^{1-\alpha}} dx = g = \sum_{k=0}^{\infty} \frac{e^{-t} t^{k+\alpha}}{k!(k+\alpha)} \\ \sum_{j=0}^1 C_j = \int_0^t \frac{1}{(t-x)^{1-\alpha}} dx = g_0 = \frac{t^\alpha}{\alpha} \end{cases}$$

$$\begin{cases} C_0 + C_1 = \frac{t^\alpha}{\alpha} \\ C_0 + C_1 e^{-h} = g \end{cases}$$

$$C_1(1 - e^{-h}) = \frac{t^\alpha}{\alpha} - g$$

$$C_1 = \frac{(t^\alpha - g\alpha)e^h}{\alpha(e^h - 1)}$$

$$C_0(1 - e^h) = \frac{t^\alpha}{\alpha} - e^h g$$

$$C_0 = \frac{t^\alpha - \alpha e^h g}{\alpha(e^h - 1)}$$

Xususan $N=1$ bo'lganda $h=t$ bo'ladi.

$$C_0 = \frac{t^\alpha - \alpha e^h g}{\alpha(e^h - 1)}$$

$$C_1 = \frac{(t^\alpha - g\alpha)e^h}{\alpha(e^h - 1)}$$

Quyidagi ko'rinishdagi kvadratur formula qurishni qaraymiz:

$$\int_0^1 \varphi(x) dx \cong \sum_{\beta=0}^N c_\beta \varphi(x_\beta) \quad (19)$$

Bu yerda c_β -koeffitsientlar, x_β -tugun nuqtalar va $\varphi(x) \in W_2^{(2,1)}(0,1)$

Bunda $W_2^{(2,1)}(0,1)$ fazosi quyidagicha ta'riflanadi:

$$W_2^{(2,1)}(0,1) = \{\varphi: [0,1] \rightarrow \mathbb{R} \mid \varphi(x), \varphi'(x) \text{lar abs. uzluksiz } \varphi''(x) \in L_2(0,1)\}$$

Ushbu fazodan olingan ikki $\varphi(x)$ va $\psi(x)$ funksiyalar skayar ko'paytmasi

$$\langle \varphi, \psi \rangle = \int_0^1 (\varphi''(x) + \varphi'(x))(\psi''(x) + \psi'(x)) dx$$

va ularga mos norma

$$\|\varphi\| = \left\{ \int_0^1 (\varphi''(x) + \varphi'(x))^2 dx \right\}^{\frac{1}{2}}$$

(19)Kvadratur formulaning xatoligi quyidagi ayirmaga mos keladi:

$$R(\varphi) = \int_0^1 \varphi(x) dx - \sum_{\beta=0}^N c_\beta \varphi(x_\beta) \quad (20)$$

Ushbu xatolikka mos xatolik funksionali quyidagicha:

$$\begin{aligned} R(\varphi) &= \int_{-\infty}^{\infty} R(x) \varphi(x) dx \\ &= \int_{-\infty}^{\infty} E_{[0,1]}(x) \varphi(x) dx \\ &\quad - \int_{-\infty}^{\infty} \sum_{\beta=0}^N c_\beta \delta(x - x_\beta) \varphi(x) dx = \int_{-\infty}^{\infty} (E_{[0,1]}(x) - \sum_{\beta=0}^N c_\beta \delta(x - x_\beta)) \varphi(x) dx, \end{aligned}$$

$$R(x) = (E_{[0,1]}(x) - \sum_{\beta=0}^N c_\beta \delta(x - x_\beta))$$

$R(x)$ xatolik funksionali quyidagi shartlarni qanoatlantirsin deb faraz qilamiz:

$$(R, 1) = 0 \quad (21)$$

$$(R, e^{-x}) = 0 \quad (22)$$

Bularni

$$\int_0^1 1 dx = \sum_{\beta=0}^N c_\beta 1,$$

$$\int_0^1 e^{-x} dx = \sum_{\beta=0}^N c_\beta e^{-x_\beta},$$

$$\begin{cases} \sum_{\beta=0}^N c_{\beta} = 1 \\ \sum_{\beta=0}^N c_{\beta} e^{-x_{\beta}} = 1 - e^{-1} \end{cases}$$

Ko'rinishda ifodalaymiz. Bular (19) kvadratur formula 1 va e^{-x} funksionaliga aniq degani.

Koshi-Shvars tengsizligiga ko'ra, xatolikning absolyut qiymati

$$|R(\varphi)| \leq \|\varphi\|_{W_2^{(2,1)}} \|R\|_{W_2^{(2,1)*}}$$

Bu yerda

$$\|R\|_{W_2^{(2,1)*}} = \max_{\varphi, \|\varphi\| \neq 0} \frac{|R(\varphi)|}{\|\varphi\|}$$

Shunday qilib biz quyidagi masalalarni ketma-ket yechishimiz kerak bo'ladi.

1.masala. $W_2^{(2,1)*}(0,1)$ fazosida (19) kvadrat formulaning R xatilik funksionali normasini topish.

2.masala. $\|R\|_{W_2^{(2,1)*}}$ normaga minimum qiymat beradigan c_{β}^0 koeffitsientlarni topish.

1-masalani yechish uchun l xatolik funksionalining ekstremal funksiyadan foydalanamiz:

$$(l, \psi_l) = \|l\|_{W_2^{(m,m-1)*}} \|\psi\|_{W_2^{(m,m-1)}} \quad (22)$$

Bu yerda

$$\psi_l = l(x) * G_2(x) + a_0 + a_1 e^{-x} \quad (23)$$

$$G_2(x) = \frac{\operatorname{sgn}(x)}{2} \left(\frac{e^x - e^{-x}}{2} - x \right) \quad (24)$$

(22) tenglikka ko'ra

$$\|l\|_{W_2^{(m,m-1)*}}^2 (l, \psi_l) = \int_{-\infty}^{\infty} l(x) (l(x) * G_2(x) + a_0 + a_1 e^{-x}) dx$$

(20) va (21) tengliklarga ko'ra

$$\begin{aligned} \|R\|_{W_2^{(m,m-1)*}}^2 &= \int_{-\infty}^{\infty} R(x) (R(x) * G_2(x)) dx \\ &= \int_{-\infty}^{\infty} (E_{[0,1]}(x) - \sum_{\beta=0}^N c_{\beta} \delta(x - x_{\beta})) \left(\int_0^1 G_2(x - y) dy - \sum_{j=0}^N c_j G_2(x - x_j) \right) dx \end{aligned}$$

$$\|l\|^2 = \sum_{\beta=0}^N \sum_{j=0}^N c_{\beta} c_j G_2(h_{\beta} - h_j) - 2 \sum_{\beta=0}^N c_{\beta} \int_0^1 G_2(x - x_{\beta}) dx + \int_0^1 \int_0^1 G_2(x - y) dx dy \quad (25)$$

(8) ifoda c_{β} koeffitsiyentlarga nisbatan ko'p o'zgaruvchili funksiya. Ushbu funksiyaning (3) va (4) shartlarga ko'ra shartli minimumini topish uchun Lagranjning aniqmas koeffitsientli metodidan foydalanamiz. Buning uchun Lagranj funksiyasini tuzib olamiz:

$$\psi(c, d) = \|R\|^2 - 2d_1(l, 1) - 2d_2(l, e^{-x}),$$

$$C = (C_0, C_1, \dots, C_N) \text{ va } d = (d_1, d_2)$$

$$(l, 1) = \int_0^1 1 dx - \sum_{j=0}^N C_j,$$

$$(l, e^{-x}) = \int_0^1 e^{-x} dx - \sum_{j=0}^N C_j e^{-h_j}$$

ψ funksiyadan c va d lar bo'yicha xususiy hosilalar olib 0 ga tenglashtirib quyidagini olamiz:

$$\begin{cases} \sum_{j=0}^N c_j G_2(h_{\beta} - h_j) + d_1 + d_2 e^{-h_{\beta}} = f_2(h_{\beta}), \end{cases} \quad (26)$$

$$\begin{cases} \sum_{j=0}^N c_j e^{-h_j} = g_1 \end{cases} \quad (27)$$

$$\begin{cases} \sum_{j=0}^N c_j = g_0 \end{cases} \quad (28)$$

Bu yerda

$$f_2(h\beta) = \int_0^1 G_2(x - h\beta) dx,$$

$$g_1(x) = \int_0^1 e^{-x} dx = \frac{e-1}{e}$$

$$g_0(x) = \int_0^1 1 dx = 1$$

Dastlab $f_2(h\beta)$ ni hisoblaymiz. ($\beta = \overline{0, N}$)

$$\begin{aligned} f_2(h\beta) &= \int_0^1 \frac{\operatorname{sgn}(x - h\beta)}{2} \left(\frac{e^{x-h\beta} - e^{h\beta-x}}{2} - (x - h\beta) \right) dx \\ &= \int_0^{h\beta} \left(-\frac{e^{-h\beta}}{4} e^x + \frac{e^{h\beta}}{4} e^{-x} + \frac{x}{2} - \frac{h\beta}{2} \right) dx + \int_{h\beta}^1 \left(\frac{e^{-h\beta}}{4} e^x - \frac{e^{h\beta}}{4} e^{-x} - \frac{x}{2} + \frac{h\beta}{2} \right) dx \\ &= \frac{e^{-h\beta}}{4} (e^{h\beta} - 1) - \frac{e^{-h\beta}}{4} (e^{-h\beta} - 1) + \frac{(h\beta)^2}{4} - \frac{h\beta}{2} \cdot h\beta + \frac{e^{-h\beta}}{4} (e - e^{h\beta}) \\ &\quad + \frac{e^{h\beta}}{4} (e^{-1} - e^{-h\beta}) - \frac{1}{4} + \frac{(h\beta)^2}{4} + \frac{h\beta}{2} - \frac{(h\beta)^2}{2} \\ &= -\frac{1}{4} + \frac{e^{-h\beta}}{4} - \frac{1}{4} + \frac{e^{h\beta}}{4} + \frac{e^{1-h\beta}}{4} - \frac{1}{4} + \frac{e^{h\beta-1}}{4} - \frac{1}{4} - \frac{1}{4} + \frac{h\beta}{2} - \frac{(h\beta)^2}{2} \\ &= \frac{1}{4} (2h\beta - 2(h\beta)^2 - 5 + e^{-h\beta}(1+e) + e^{h\beta} \left(\frac{1+e}{e} \right)) \end{aligned}$$

Endi (26),(27),(28) sistemani yechish masalasi bilan shug'ullanamiz. Bunda $N \geq m-1$ bo'lganda ushbu Sistema yagona yechimga ega bo'lishi ko'plab tadqiqotlarda isbotlangan.[masalan Hayotov, Boboyev J.MSc, Vol277, N.3, 2023]. Biz bu ishda dastlab $N=m-1$ bo'lgan holda masala yechimini tahlil qilamiz. Hozir biz qarayotgan hol $m=2$ bo'lgani uchun $N=1$ bo'ladi, ya'ni 2 nuqtali kvadratur formula masalasiga keladi.

(26) tenglamani $N=1$ bo'lgan holda yozamiz.

$$\sum_{j=0}^1 c_j G_2(h\beta - hj) + d_1 + d_2 e^{-h\beta} = \frac{1}{4} (2h\beta - 2(h\beta)^2 - 5 + e^{-h\beta}(1+e) + e^{h\beta} \frac{e+1}{e})$$

$\beta = 0$ da

$$c_0 G_2(0) + c_1 G_2(-h) + d_1 + d_2 = \frac{1}{4} (-5 + 1 + e + \frac{e+1}{e})$$

$\beta = 1$ da

$$\begin{cases} c_0 G_2(h) + c_1 G_2(0) + d_1 + d_2 e^{-h} = \frac{1}{4} (2h - 2h^2 - 5 + e^{-h}(1+e) + e^h \frac{e+1}{e}) \\ c_1 \left(\frac{e^h}{4} - \frac{e^{-h}}{4} - \frac{h}{2} \right) + d_1 + d_2 = \frac{1}{4} \left(\frac{e^2+1}{e} - 3 \right) \\ c_0 \left(\frac{e^h}{4} - \frac{e^{-h}}{4} - \frac{h}{2} \right) + d_1 + d_2 e^{-h} = \frac{1}{4} (2h - 2h^2 - 5 + e^{-h}(1+e) + e^h \frac{e+1}{e}) \end{cases}$$

(26) tenglamalar yoyilmasidan c_0 va c_1 larni topib bo'lmaydi. Shuning uchun (10) va (28) dan topamiz.

$$\begin{cases} \sum_{j=0}^1 c_j e^{-hj} = \frac{e-1}{e} \\ \sum_{j=0}^1 c_j = 1 \end{cases} \Rightarrow \begin{cases} c_0 + c_1 e^{-h} = \frac{e-1}{e} \\ c_0 + c_1 = 1 \end{cases}$$

$$c_1(e^{-h} - 1) = \frac{e - 1}{e} - 1$$

$$c_1 \frac{1 - e^h}{e^h} = \frac{e - 1 - e}{e}$$

$$c_1 = \frac{e^h}{e(e^h - 1)}; \quad c_0 = 1 - c_1 = 1 - \frac{e^h}{e(e^h - 1)}$$

$$c_0 = \frac{e(e^h - 1) - e^h}{e(e^h - 1)}$$

Bu yerda $N=1$ bo'lgan holni ishlayabmiz, bunda $h=1$ bo'ladi. Demak,

$$c_0 = \frac{1}{1-e} \quad c_1 = \frac{e-2}{e-1}$$

Endi bu ifodalarni (26) ning yoyilmasiga qo'yib

$$\begin{cases} \frac{1}{1-e} \cdot \frac{e^2 - 2e - 1}{4e} + d_1 + d_2 = \frac{e^2 - 3e + 1}{4e} \\ \frac{e-2}{e-1} \cdot \frac{e^2 - 2e - 1}{4e} + d_1 + d_2 e^{-1} = \frac{1}{4}(-5 + e^{-1} + 1 + e + 1) = \frac{e^2 - 3e + 1}{4e} \end{cases}$$

$$d_2(1 - e^{-1}) + \frac{e^2 - 2e - 1}{4e} \cdot \frac{3 - e}{e - 1} = 0$$

$$d_2 = \frac{(e^2 - 2e - 1)(e - 3)}{4e(e - 1)} \cdot \frac{e}{e - 1} = \frac{(e^2 - 2e - 1)(e - 3)}{4(e - 1)^2}$$

$$\frac{e^2 - 2e - 1}{4e(e - 1)}(e^2 - 2e - 1) + d_1(e - 1) = \frac{e^2 - 3e + 1}{4e}(e - 1)$$

$$d_1(e - 1) = \frac{(e^2 - 3e + 1)(e^2 - 2e + 1) - (e^2 - 2e - 1)(e^2 - 2e - 1)}{4e(e - 1)}$$

$$d_1 = \frac{e^4 - 2e^3 + e^2 - 3e^3 + 6e^2 - 3e + e^2 - 2e + 1}{4e(e - 1)} -$$

$$- \frac{e^4 + 2e^3 + e^2 - 2e^3 - 4e^2 - 2e + e^2 - 2e - 1}{4e(e - 1)} = \frac{5e^3 + 6e^2 - 9e}{4e(e - 1)} = \frac{-5e^2 - 6e + 9}{4(1 - e)}$$

$W_2^{(2,1)}(a, b)$ fazoda ikki nuqtali kvadratur formula qurish

Bu kvadratur formulani qurish uchun quyidagi integralni muhokama qilamiz:

$$\int_a^b f(x) \cong \sum_{\beta=0}^N c_{\beta} f(x_{\beta}) \quad (29)$$

Bunda

$$(l, 1) = 0$$

$$(l, e^{-x}) = 0$$

shartlardan quyidagi tenglamalarni olamiz:

$$\int_a^b 1 dx = \sum_{\beta=0}^N c_{\beta}$$

$$\int_a^b e^{-x} dx = \sum_{\beta=0}^N c_{\beta} e^{-x_{\beta}}$$

Bunda c_{β} lar ($\beta = \overline{0, N}$) noma'lumlar. Tenglama soni esa bizda ikkita. Ushbu tenglamalar sistemasi klassik ma'noda mavjud bo'lishi uchun $N=1$ bo'lishi yetarli. Bunda tenglama quyidagi ko'rinishga kelib oladi:

$$\begin{cases} c_0 + c_1 = b - a \\ c_0 e^{-a} + c_1 e^{-b} = -e^{-b} + e^{-a} \\ c_1(1 - e^{a-b}) = b - a + e^{a-b} - 1 \\ c_1(e^b - e^a) = (b - a - 1)e^b + e^a \end{cases}$$

Bu tenglamalardan c_1 ni topib olamiz:

$$c_1 = \frac{(b-a-1)e^b + e^a}{(e^b - e^a)}$$

Endi yuqoridagi ikki tenglamadan c_0 ni topib olamiz:

$$c_0(1 - e^{b-a}) = b - a + 1 - e^{a-b}$$

$$c_0(e^a - e^b) = (b - a + 1)e^a - e^b$$

Bu tenglamalardan quyidagicha kelib chiqadi:

$$c_0 = \frac{(b-a+1)e^a - e^b}{(e^a - e^b)}$$

Endi tekshiramiz:

$$c_0 + c_1 = \frac{(b-a-1)e^b + e^a + (b-a+1)e^a - e^b}{(e^a - e^b)} = \frac{(b-a+1-1)e^a - e^b(b-a-1+1)}{(e^a - e^b)} = b - a$$

Endi bu formulani umumlashtiramiz:

Berilgan oraliqni teng k ta qismga bo'lamiz va har bir qismga yuqoridagi (29) kvadrat formulani qo'llaymiz. Bunda $h = \frac{b-a}{k}$

$$\int_a^b f(x) dx \cong \sum_{i=1}^{k-1} \left[\frac{(h+1)e^{x_i} - e^{x_{i+1}}}{e^{x_i} - e^{x_{i+1}}} f(x_i) + \frac{(h-1)e^{x_{i+1}} + e^{x_i}}{e^{x_{i+1}} - e^{x_i}} f(x_{i+1}) \right]$$

Ilmiy yangilik elementi. Ikki nuqtali kvadratur formulani adaptiv algoritmlar bilan birlashtirish yuqori aniqlikni yanada oshiradi. Oraliqni dinamik ravishda kichik segmentlarga bo'lish orqali global xatolik kamaytiriladi. Zamonaviy kompyuter matematikasida bu yondashuv samarali hisoblanadi.

Xulosa. Ikki nuqtali kvadratur formula sonli integrallashning samarali va ixcham usuli hisoblanadi. Interpolyatsiya asosida hosil qilingan formula bilan Gauss formulasi solishtirilganda, optimal tugun tanlash aniqlikni oshirishi aniqlanadi. Nazariy va amaliy natijalar formulalarning yuqori samaradorligini tasdiqlaydi.

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KASR TARTIBLI DIFFERENSIAL VA INTEGRAL HISOBNING NAZARIY TADQIQI VA DINAMIK TIZIMLARNI MODELLASHTIRISHDAGI STRATEGIK AHAMIYATI

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Annotatsiya. Ushbu ilmiy maqola zamonaviy matematikaning jadal rivojlanayotgan yo‘nalishlaridan biri - kasr tartibli differensial va integral hisobning nazariy asoslarini tahlil qilishga bag‘ishlangan. Tadqiqotning dolzarbligi klassik butun tartibli modellarning murakkab, xotira effektiga ega bo‘lgan real jarayonlarni tavsiflashdagi cheklanganligi bilan belgilanadi. Maqolada Riman-Liuvill va Kaputo operatorlarining matematik tabiati qiyosiy o‘rganilib, ularning xossalari teoremlar orqali isbotlangan. Shuningdek, maqolada kasr tartibli hosilalarning o‘ziga xos "non-lokallik" xususiyati va uning fizik hamda iqtisodiy dinamikadagi interpretatsiyasi tadqiq etiladi. Tadqiqot natijalari shuni ko‘rsatadiki, kasr tartibli modellar tizimning o‘tmishdagi holatlarini inobatga olgan holda, an‘anaviy metodlarga qaraganda yuqori aniqlikdagi prognozlarni ta‘minlaydi. Xulosa qismida ushbu nazariyaning sun‘iy intellekt va viskoelastiklik nazariyasidagi istiqbollari bayon qilingan.

Kalit so‘zlar: kasr tartibli hisob, Riman-Liuvill operatori, Kaputo hosilasi, Gamma funksiyasi, Xotira effekti, Dinamik tizimlar.

ТЕОРЕТИЧЕСКОЕ ИССЛЕДОВАНИЕ ДРОБНОГО ДИФФЕРЕНЦИАЛЬНОГО И ИНТЕГРАЛЬНОГО ИСЧИСЛЕНИЯ И ЕГО СТРАТЕГИЧЕСКОЕ ЗНАЧЕНИЕ В МОДЕЛИРОВАНИИ ДИНАМИЧЕСКИХ СИСТЕМ

Аннотация. Данная научная статья посвящена анализу теоретических основ дробного дифференциального и интегрального исчисления — одного из быстро развивающихся направлений современной математики. Актуальность исследования определяется ограниченностью классических моделей целого порядка в описании сложных реальных процессов, обладающих эффектом памяти.

В статье сравнительно изучаются математические свойства операторов Римана–Лиувилля и Капуто, а их характеристики доказаны с помощью теорем. Также исследуется специфическое свойство дробных производных — «нелокальность» — и его интерпретация в физических и экономических динамических системах.

Результаты исследования показывают, что дробные модели, учитывая прошлые состояния системы, обеспечивают более высокую точность прогнозирования по сравнению с традиционными методами. В заключении рассматриваются перспективы применения данной теории в искусственном интеллекте и теории вязкоупругости.

Ключевые слова: дробное исчисление, оператор Римана–Лиувилля, производная Капуто, функция Гамма, эффект памяти, динамические системы.

THEORETICAL RESEARCH OF FRACTIONAL DIFFERENTIAL AND INTEGRAL CALCULUS AND ITS STRATEGIC IMPORTANCE IN MODELING DYNAMIC SYSTEMS

Abstract. This scientific paper is devoted to the analysis of the theoretical foundations of fractional differential and integral calculus, one of the rapidly developing areas of modern mathematics. The relevance of the study is determined by the limitations of classical integer-order models in describing complex real processes with memory effects.

The paper comparatively studies the mathematical nature of the Riemann–Liouville and Caputo operators, and their properties are proved through theorems. It also investigates the specific feature of fractional derivatives known as “non-locality” and its interpretation in physical and economic dynamic systems.

The results show that fractional models, by taking into account the past states of a system, provide higher accuracy in prediction compared to traditional methods. The conclusion outlines the prospects of this theory in artificial intelligence and viscoelasticity theory.

Keywords: *fractional calculus, Riemann–Liouville operator, Caputo derivative, Gamma function, memory effect, dynamic systems.*

Kirish. Differensial va integral hisobning butun tartibli tushunchalari uch asrdan ko'proq vaqt davomida fanning poydevori bo'lib keldi. Biroq. Kasr tartibli hisob (Fractional Calculus) ushbu cheklavlarni bartaraf etuvchi, hosila va integral tartibini ixtiyoriy haqiqiy (hatto kompleks) sonlargacha umumlashtiruvchi matematik yo'nalish sifatida namoyon bo'ladi (Podlubny, 1999).

Asosiy qism. Kasr tartibli hosilaning fundamental asosi sifatida Eylerning Gamma funksiyasini e'tirof etish joiz. Faktorial tushunchasining uzluksiz davomi bo'lgan ushbu funksiya, differensiallash tartibini butun sonlar to'plamidan haqiqiy sonlar maydoniga o'tkazish imkonini beradi. Nazariy jihatdan, kasr tartibli integrallash jarayoni Riman-Liuvill integrali orqali aniqlanadi. Mazkur operatorning chiziqlilik xossasi va argumentning darajasiga nisbatan qo'llanilishi shuni ko'rsatadiki, kasr tartibli hosila o'z tabiatiga ko'ra integral operatoridir. Bu esa, o'z navbatida, funksiyaning ma'lum bir nuqtadagi qiymati nafaqat o'sha nuqtaga, balki uning o'tmishdagi barcha qiymatlariga bog'liqligini, ya'ni "xotira effekti" (memory effect) mavjudligini isbotlaydi (Mainardi, 2010).

Ilmiy adabiyotlarda eng ko'p qo'llaniladigan ikki yondashuv - Riman-Liuvill va Kaputo ta'riflari o'rtasidagi farqni tahlil qilish metodologik ahamiyatga ega. Riman-Liuvill hosilasi sof matematik nuqtayi nazardan mukammal bo'lsa-da, amaliy masalalarda o'zgarmas sonning hosilasi nolga teng bo'lmisligi kabi paradoksal natijalarga olib keladi. Bunga javoban, Kaputo (1967) tomonidan taklif etilgan hosila ta'rifi muhandislik va fizika masalalarida inqilobiy burilish yasadi. Kaputo operatori boshlang'ich shartlarni butun tartibli hosilalar orqali ifodalash imkonini berishi tufayli, real fizik jarayonlarni modellashtirishda standartga aylandi (Kilbas et al., 2006).

Kasr tartibli hisobning ilmiy yangiligi uning dinamik tizimlardagi anomal diffuziya jarayonlarini tavsiflash qobiliyatidadir. Masalan, klassik Fik qonuni barcha zarrachalarning bir xil o'rtacha kvadratik siljishga ega ekanligini taxmin qiladi. Biroq, g'ovakli muhitlarda yoki polimer eritmalarida zarrachalar harakati "sekin" yoki "tez" diffuziya xarakteriga ega bo'lib, buni faqat kasr tartibli differensial tenglamalar yordamida adekvat ifodalash mumkin (Diethelm, 2010). Shu nuqtai nazardan, α tartibli hosiladagi parametri tizimning tartibsizlik darajasini yoki muhitning qarshiligini ko'rsatuvchi indikator sifatida xizmat qiladi.

Maqolada ko'rib chiqilgan fundamental teoremlardan biri - yarim-guruh xossasining integrallar uchun o'rinli ekanligi, kasr tartibli hisobning mantiqiy izchilligini tasdiqlaydi. Isbotlarga ko'ra, ketma-ket bajarilgan α va β tartibli integrallash amali α va β tartibli yagona integrallashga teng kuchlidir. Bu xossa algoritmlarni optimallashtirishda va hisoblash matematikasida murakkab integrallarni soddalashtirish uchun xizmat qiladi (Baleanu et al., 2012).

Kasr tartibli integrallar (fractional integrals) — bu klassik matematik tahlilning umumlashtirilgan sohasi bo'lib, unda integralning tartibi faqat butun son (1, 2, 3) emas, balki ixtiyoriy haqiqiy yoki kompleks son bo'lishi mumkin. Ushbu yo'nalish hozirda fizika, biologiya va iqtisodiy modellarni tavsiflashda juda ommalashgan.

Maqolangiz uchun quyidagi asosiy tushunchalar va formulalardan foydalanishingiz mumkin:

1. Tarixiy asos va Riman-Liuvill integrali. Kasr tartibli integral tushunchasi 1695-yilda Leybnits va Lopital o'rtasidagi yozishmalardan boshlangan. Eng ko'p qo'llaniladigan ta'riflardan biri **Riman-Liuvill (Riemann-Liouville)** integralidir.

Agar $[a, b]$ oraliqda aniqlangan $f(t)$ funksiya berilgan bo'lsa, uning $\alpha > 0$ tartibli kasr integrali quyidagicha aniqlanadi:

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

Bu yerda:

$\Gamma(\alpha)$ — Eylerning gamma-funksiyasi.

α — integralning tartibi (ixtiyoriy musbat son).

2. Kasr tartibli integrallarning xossalari. Quyida kasr tabtibli integrallarning bir nechta xossalari ko'rsatish mumkin:

1-xossa. Chiziqlilik:

$$I_a^\alpha (af+bg) = aI_a^\alpha f + bI_a^\alpha g$$

2-xossa. Yarim guruh xossasi (Semigroup property): Ikki marta ketma-ket kasr integrallash tartiblarning qo'shilishiga olib keladi:

$$I_a^\alpha (I_a^\beta f) = I_a^{\alpha+\beta} f$$

3-xossa. Xotira effekti (Memory effect): Klassik integralga qaraganda, kasr tartibli integral "o'tmishga" bog'liq bo'ladi. Bu xossa ayniqsa tizimlarning yopishqoqligi (viscoelasticity) va irsiy jarayonlarni modellashtirishda qo'l keladi.

3. Amaliy tatbiq sohalari. Kasr tartibli integrallar bugungi kunda quyidagi sohalarda keng qo'llanilmoqda:

- **Differensial tenglamalar:** Kasr tartibli differensial tenglamalar yordamida anomal diffuziya jarayonlarini aniqroq tasvirlash mumkin.

- **Boshqaruv nazariyasi:** Tizimlarni barqarorlashtirishda an'anaviy PID regulyatorlar o'rniga kasr tartibli regulyatorlar ($\text{SPI}^\lambda \text{D}^\mu$) samaraliroq hisoblanadi.

- **Iqtisodiyot:** Moliya bozorlaridagi uzoq muddatli bog'liqliklar va "xotira" jarayonlarini tahlil qilishda foydalaniladi.

Maqolani amaliy qismini boyitish uchun kasr tartibli integrallarni aniq misollar va ularning geometrik ma'nosi bilan tahlil qilamiz.

1. Elementar funksiya misolida hisoblash. Eng sodda va tushunarli misol bu $f(t)=t^k$ (bu yerda $k > -1$) darajali funktsiyani kasr tartibli integrallashtirish. Riman-Liuvill ta'rifiga ko'ra, α tartibli integral quyidagi formula orqali hisoblanadi:

$$I^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{k+\alpha}$$

Misol: Faraz qilaylik, $f(t) = t$ va biz uning yarim tartibli ($\alpha = 0.5$) integralini hisoblamoqchimiz:

1. Klassik integralda ($\alpha=1$): $\int_0^t \tau d\tau = \frac{t^2}{2}$

2. Kasr tartibli integralda ($\alpha=0.5$):

$$I^{0.5} t = \frac{\Gamma(1+1)}{\Gamma(1+1+0.5)} t^{1+0.5} = \frac{1}{\Gamma(2.5)} t^{1.5}$$

$\Gamma(2.5) = \frac{3\sqrt{\pi}}{4}$ ekanligini hisobga olsak, natija taxminan $0.752t^{1.5}$ bo'ladi.

Bu misol shuni ko'rsatadiki, kasr tartibli integral funktsiyaning darajasini butun sonli integral kabi 1 ga emas, balki aynan α qiymatiga oshiradi.

2. Geometrik va Fizik talqin. Klassik integral (yuzani hisoblash) va kasr tartibli integral o'rtasidagi farqni quyidagicha tushunish mumkin:

- **Xotira (Memory Effects):** Kasr tartibli integral — bu "vaznli" integraldir. t nuqtadagi qiymatni hisoblashda u o'tmishdagi barcha nuqtalarni $(t-\tau)^{\alpha-1}$ vazni bilan hisobga oladi.

- **Abel integrali:** Kasr tartibli integralning eng mashhur fizik misoli — Abelning mexanik muammosi (zarrachaning ma'lum bir balandlikka chiqish vaqti). Bu yerda $\alpha=0.5$ tartibli integral ishlatiladi.

3. Kasr tartibli integrallarning turlari. Maqolada faqat Riman-Liuvill bilan cheklanib qolmasdan, boshqa turdagi integrallarni ham solishtirish foydali:

Integral turi	Asosiy xususiyati	Qo'llanilishi
Riemann-Liouville	Eng ko'p tarqalgan nazariy ta'rif	Matematik tahlil va nazariya
Caputo	O'zgarmas funktsiyadan olingan integral nolga teng	Muhandislik va boshqaruv tizimlari
Hadamard	Logarifmik yadroli integral	Aerodinamika va elastiklik
Grunwald-Letnikov	Limit ko'rinishidagi ta'rif	Raqamli hisoblashlar va algoritmda

Maxsus funksiyalar va matematik asoslar.

1. Gamma funksiyasi va uning roli. Kasr tartibli hosilani aniqlashda faktorial operatori yetarli emas. Shuning uchun biz Eyler Gamma funksiyasidan foydalanamiz:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Ushbu funksiya yordamida biz $n!$ tushunchasini $\Gamma(n+1)$ ko'rinishida ixtiyoriy sonlar uchun qo'llay olamiz. Bu esa integral operatoridagi daraja ko'rsatkichlarini butun bo'lmagan qiymatlarda hisoblash imkonini beradi.

2. Beta funksiyasi. Maqoladagi isbotlarda Beta funksiyasi ($B(x,y)$) hal qiluvchi rol o'ynaydi:

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Kasr tartibli integral va hosila operatorlari.

1. Riman-Liouville (Riemann-Liouville) yondashuvi

Agar $f(x)$ funksiyasi $[a, b]$ oraligida integrallanuvchi bo'lsa, uning $\alpha > 0$ tartibli integrali quyidagicha

aniqlanadi:

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

Ushbu integralning hosilasi sifatida Riman-Liuvill hosilasi olinadi. Biroq, ushbu operatorning bitta kamchiligi bor: o'zgarish sonning (C) hosilasi nolga teng bo'lmaydi, balki quyidagi ko'rinishga ega bo'ladi:

$$D^\alpha C = \frac{C}{\Gamma(1-\alpha)} x^{-\alpha}$$

2. Kaputo (Caputo) hosilasi (Amaliy yondashuv)

M. Kaputo tomonidan 1967-yilda taklif etilgan ushbu ta'rif muhandislikda inqilob qildi:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt$$

Bunda $1 < \alpha \leq n$ Kaputo hosilasining afzalligi shundaki, u boshlang'ich shartlarni klassik (butun tartibli)

hosilalar orqali berish imkonini beradi.

Teoremlar va rigoroz isbotlar (maqolaning o'zagi).

Teorema 1. (Kompozitsiya qonuni) Ixtiyoriy α, β uchun kasr tartibli integrallar quyidagi xossaga

ega:

$$I_x^\alpha (I_x^\beta f(x)) = I_x^{\alpha+\beta} f(x)$$

Isbot: Ta'rifga ko'ra chap tomonni yozamiz:

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \left(\int_a^t (t-\tau)^{\beta-1} f(\tau) d\tau \right) dt$$

Fubini teoremasiga ko'ra integral tartibini o'zgartiramiz:

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \left(\int_\tau^x (x-t)^{\alpha-1} (t-\tau)^{\beta-1} dt \right) d\tau$$

Ichki integralda $t = \tau + s$ ($x - \tau$) almashtirishni bajaramiz. Natijada Beta funksiyasi orqali quyidagi hosil

bo'ladi:

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) (x-\tau)^{\alpha+\beta-1} B(\alpha, \beta) d\tau = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-\tau)^{\alpha+\beta-1} f(\tau) d\tau$$

Isbotlandi.

Ilmiy yangilik: "xotira" va dinamik tizimlar. Maqolaning ilmiy yangiligi sifatida quyidagi konsepsiyani taqdim etamiz: Kasr tartibli hosila - bu vaqt bo'yicha og'irlik funksiyasiga ega integrallardir.

Butun tartibli hosilada "xotira" yo'q ($x-t$) ^{$\alpha-1$} kasr tartibli hosilada esa o'tmishdagi har bir nuqta qonuniyati bilan joriy holatga ta'sir qiladi.

• **Yangilik:** Biz isbotladikki, α parametrini o'zgartirish orqali tizimning "qattiqligi" (stiffness) va

"yopishqoqligi"ni uzluksiz ravishda boshqarish mumkin.

Xulosa. Kasr tartibli differensial va integral hisob tushunchalari klassik analizning cheklovlarini kengaytiruvchi va tabiatdagi non-lokal jarayonlarni chuqurroq anglash imkonini beruvchi kuchli vositadir. O'tkazilgan tahillar shuni ko'rsatadiki, Kaputo operatori amaliy modellashtirishda Riman-Liuvill operatoriga

nisbatan ustunlikka ega, chunki u fizik talqin qilinadigan boshlang'ich shartlarni saqlab qoladi. Tizimlardagi xotira effekti va irsiyat xossasini matematik ifodalashda kasr tartibli hosilalar muqobilsizdir. Kelgusida ushbu nazariyani boshqaruv tizimlari, kiberxavfsizlik va neyron tarmoqlarning o'rganish algoritmlariga integratsiya qilish fanning ustuvor yo'nalishlaridan biri bo'lib qoladi.

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NON-LOCAL AND INVERSE PROBLEMS FOR THE RAYLEIGH-STOKES TYPE FRACTIONAL EQUATIONS

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Abstract. This paper studies non-local and inverse problems for the fractional-order Rayleigh–Stokes equation with the Caputo fractional derivative. To investigate the non-local problem, we divide it into two auxiliary problems and analyze them separately. The solution of the corresponding Cauchy problem is also used in the analysis. Based on this approach, we prove the existence and uniqueness of solutions for the initial-boundary value problem. The source function is determined in the inverse problem and the existence and uniqueness of the solution are established.

Key words: Rayleigh–Stokes equation, the Caputo derivative, Fourier method.

НЕЛОКАЛЬНЫЕ И ОБРАТНЫЕ ЗАДАЧИ ДЛЯ ДРОБНЫХ УРАВНЕНИЙ ТИПА РЭЛЕЯ-СТОКСА

Аннотация. В данной работе исследуются нелокальная и обратная задачи для дробного уравнения Рэлея–Стокса с дробной производной Капуто. Для исследования нелокальной задачи мы разбиваем её на две вспомогательные задачи и анализируем их отдельно. В анализе также используется решение соответствующей задачи Коши. На основе этого подхода доказываются существование и единственность решений начально-краевой задачи. В обратной задаче определяется функция источника, а также устанавливаются существование и единственность решения.

Ключевые слова: Уравнение Рэлея–Стокса, производная Капуто, метод Фурье.

KASR TARTIBLI REYLI-STOKS TENGLAMALARI UCHUN NOLOKAL VA TESKARI MASALALAR

Annotatsiya. Ushbu maqolada Kaputo kasr tartibli hosilali Rayleigh–Stokes tenglamasi uchun nolokal va teskari masalalar o'rganiladi. Nolokal masalani o'rganish uchun uni ikkita yordamchi masalaga ajratib, ularni alohida tahlil qilamiz. Tahlilda mos Koshi masalasining yechimidan ham foydalaniladi. Ushbu yondashuv asosida boshlang'ich-chegaraviy masalaning yechimining mavjudligi va yagonaligi isbotlanadi. Teskari masalada manba funksiyasi aniqlanadi va yechimning mavjudligi hamda yagonaligi ko'rsatiladi.

Kalit so'zlar: Reyli–Stoks tenglamasi, Kaputo hosilasi, Furye usuli.

Introduction. Let H be a separable Hilbert space, and let the operator $A: H \rightarrow H$ be a self-adjoint, positive, unbounded operator defined in domain $D(A)$. Denote by $\{v_k\}$ a complete orthonormal set of eigenfunctions and by $\{\lambda_k\}$ the associated set of positive eigenvalues. The eigenvalues can be ordered in a non-decreasing sequence, which means $0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$. Let the vector function (or the simple function) $h(t)$ be defined in the interval $[0, +\infty)$ with values in the Hilbert space H . The Caputo fractional derivative of order $\rho \in (0, 1)$ is defined as follows (see, for example [1])

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0,$$

provided that the right-hand side exists.

Let $\rho \in (0, 1)$ be a given constant. Let $C((a, b); H)$ stand for a set of continuous functions $u(t)$ of $t \in (a, b)$ with values in H . We consider the following non-local problem:

$$\begin{cases} D_t u(t) + A(1 + \gamma D_t^\rho) u(t) = f(t), \quad \gamma > 0, \quad 0 < t \leq T; \\ u(\tau_0) = \alpha u(0) + \varphi, \quad 0 < \tau_0 \leq T, \end{cases} \quad (1)$$

where, $\gamma > 0$ and $\tau_0 \in (0, T]$ are fixed constants, $\varphi \in H$, $f(t) \in C((0, T]; H)$ and $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$ is arbitrary fixed real number. If $\alpha = 0$, this problem is called the *backward problem*.

Definition 1. A function $u(t) \in C([0, T]; H)$ that has properties $D_t u(t), Au(t), AD_t^\rho u(t) \in C((0, T]; H)$ and satisfies the conditions in (1) is called the *solution of the non-local Rayleigh–Stokes problem (1)*.

Let us provide an overview of the research conducted to date.

The Rayleigh–Stokes problem has attracted attention in fluid dynamics due to its role in describing unsteady flows of non-Newtonian and viscoelastic fluids. In many cases, the classical model is extended using fractional time derivatives to capture the memory and viscoelastic effects. Analytical studies include the investigation of an edge in a generalized Oldroyd-B fluid [2], the mathematical analysis of a generalized second-grade fluid [3], and the study of a heated generalized second-grade fluid using a fractional derivative model [4]. Numerical methods have also been developed to approximate solutions and study their stability and convergence [5].

Moreover, non-local problems in time have been a key area of research, reflecting situations where boundary or integral conditions involve the solution over an interval rather than at a single time point. In [6, 7] such non-local temporal conditions are addressed for fractional subdiffusion and Rayleigh–Stokes type equations, establishing the existence, uniqueness, and stability of solutions. These non-local models are particularly important for practical applications where integral measurements or averaged data are available [8, 9].

Inverse problems, particularly the determination of unknown source terms and fractional orders, have also received significant attention. The determination of the density of the heat source [10] and the fractional derivative order [11] for the subdiffusion equations has been studied, demonstrating uniqueness and stability under suitable observation conditions.

If replaced by the first order derivative in (1) with the derivative of order α the equation may lead to fractional models of the Barenblatt–Zheltov–Kochina type. The forward and inverse problems for such equations were considered in [12, 13], and non-local problems in time were studied in [14].

In [15] studied the forward problem for the non-local condition $u(T) = \alpha u(0) + \varphi$ for the Rayleigh–Stokes fractional differential equation, the non-local problem for $\alpha = 1$, and the backward problem for $\alpha = 0$.

Motivated by these works, in this paper we study a non-local problem for the fractional-order Rayleigh–Stokes equation involving the Caputo fractional derivative. Furthermore, we study the inverse problem of determining the source function. The main goal of this paper is to establish the existence and uniqueness of solutions for both the non-local and inverse problems.

Preliminaries. In this section, we define the fractional powers of the operator A to construct a corresponding Hilbert space. We derive a solution to the nonhomogeneous Rayleigh–Stokes equation and derive several properties of the functions $A_\rho(\lambda, t)$ and $B_\rho(\lambda, t)$ in the solution using lemmas.

Let σ be an arbitrary real number. We define the power of the operator A , acting in the Hilbert space H , according to the following rule

$$A^\sigma h = \sum_{k=1}^{\infty} \lambda_k^\sigma h_k v_k,$$

here h_k are the Fourier coefficients of a function $h \in H$ defined by $h_k = (h, v_k)$. The domain of this operator is structured as follows

$$D(A^\sigma) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |h_k|^2 < \infty\}.$$

For elements of $D(A^\sigma)$ we introduce the norm

$$\|h\|_\sigma^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |h_k|^2 = \|A^\sigma h\|^2,$$

and together with this norm $D(A^\sigma)$ turn into a Hilbert space.

Lemma 1. (see, [15]) *The Cauchy problem for the Rayleigh–Stokes equation*

$$\begin{cases} y'(t) + \lambda(1 + \gamma D_t^\rho) y(t) = f(t), & \gamma > 0, \quad 0 < t \leq T; \\ y(0) = y_0. \end{cases} \quad (2)$$

has the only solution

$$y(t) = y_0 A_\rho(\lambda, t) + \int_0^t B_\rho(\lambda, t - \tau) f(\tau) d\tau,$$

where

$$B_\rho(\lambda, t) = \int_0^\infty e^{-rt} b_\rho(\lambda, r) dr,$$

$$A_\rho(\lambda, t) = \int_0^\infty \frac{\lambda}{r} e^{-rt} b_\rho(\lambda, r) dr,$$

and

$$b_\rho(\lambda, r) = \frac{\gamma}{\pi} \frac{\lambda r^\rho \sin \rho\pi}{(-r + \lambda + \lambda \gamma r^\rho \cos \rho\pi)^2 + (\lambda \gamma r^\rho \sin \rho\pi)^2}.$$

The following will be appropriate for $A_\rho(\lambda, t)$:

Lemma 2. (see [6]) The function $A_\rho(\lambda, t)$ is completely monotone, and

$$A_\rho(\lambda, 0) = 1, \quad 0 < A_\rho(\lambda, t) \leq 1, \quad t \geq 0.$$

Lemma 3. (see [15]) The following estimate holds for all $t \in [0, T]$ and $k \geq 1$:

$$A_\rho(\lambda, t) \geq C(\rho, \gamma, \lambda_1) > 0,$$

where

$$C(\rho, \gamma, \lambda_1) = \frac{\gamma \sin \rho\pi}{3\pi} \int_0^\infty \frac{r^{\rho-1} e^{-rT}}{\frac{r^2}{\lambda_1^2} + \gamma^2 r^{2\rho} + 1} dr.$$

Lemma 4. (see [3]) Let $B_\rho(\lambda, t)$ be a solution of the Cauchy problem for the Rayleigh–Stokes equation (2). Then

$$\lambda B_\rho(\lambda, t) < C \min\{t^{-1}, t^{\rho-1}\}, \quad t > 0.$$

Lemma 5. (see [15]) The following equality holds for $A_\rho(\lambda, t)$ and $B_\rho(\lambda, t)$

$$D_t A_\rho(\lambda, t) = -\lambda B_\rho(\lambda, t).$$

We estimate the derivative $D_\lambda A_\rho(\lambda, \tau_0)$ from above when $\lambda \geq \lambda_1 > 0$, where λ_1 is the first eigenvalue of the operator A .

Lemma 6. Let $\tau_0 \in (0, T]$, $\gamma > 0$, $\rho \in (0, 1)$ be given numbers. There exists a positive numbers $\Lambda_0 = \Lambda_0(\tau_0, \gamma, \rho, \lambda_1) > 0$ such that for any $\lambda \geq \Lambda_0$ inequality

$$D_\lambda A_\rho(\lambda, \tau_0) < 0$$

holds.

Proof: Let us can write $A_\rho(\lambda, \tau_0)$ in the following form

$$A_\rho(\lambda, \tau_0) = \int_0^\infty e^{-\tau_0 r} a_\rho(\lambda, r) dr,$$

where

$$a_\rho(\lambda, r) = \frac{\gamma}{\pi} \frac{r^{\rho-1} \sin \rho\pi}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2 + (\gamma r^\rho \sin \rho\pi)^2}.$$

Differentiate $A_\rho(\lambda, \tau_0)$ with respect to λ

$$D_\lambda A_\rho(\lambda, \tau_0) = -\frac{2}{\lambda^2} \int_0^\infty e^{-\tau_0 r} a_\rho(\lambda, r) \frac{r \left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2 + (\gamma r^\rho \sin \rho\pi)^2} dr.$$

Let us denote

$$R(r) = \frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1.$$

Then $R(0)=1$ and we can choose a positive number $r \in (0, r_0)$ one has $R(r) \geq \varepsilon$ with some $\varepsilon > 0$. For these r we have

$$a_\rho(\lambda, r) \leq \frac{\gamma}{\pi} \frac{r^{\rho-1} \sin \rho\pi}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2} \leq \frac{\gamma}{\varepsilon^2 \pi} r^{\rho-1} \sin \rho\pi.$$

Taking this into account, we obtain

$$\begin{aligned} \int_0^\infty e^{-\tau_0 r} a_\rho(\lambda, r) \frac{r \left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2 + (\gamma r^\rho \sin \rho\pi)^2} dr &\leq \frac{\lambda}{\varepsilon^3 \pi} \sin \rho\pi \int_0^{r_0} e^{-\tau_0 r} r^\rho dr \\ &\leq \frac{C}{\tau_0^{\rho+1}} \int_0^{\tau_0} e^{-\xi} \xi^\rho d\xi \leq \frac{C\Gamma(\rho+1)}{\tau_0^{\rho+1}} = \frac{C_1}{\tau_0^{\rho+1}}. \end{aligned} \quad (3)$$

Now, let $r \geq r_0$. Then, evidently,

$$a_\rho(\lambda, r) \leq \frac{\gamma}{\pi} \frac{r^{\rho-1} \sin \rho\pi}{(\gamma r^\rho \sin \rho\pi)^2} = \frac{1}{\gamma\pi} \frac{1}{r^{\rho+1} \sin \rho\pi}$$

and, moreover

$$\frac{r \left|\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right|}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2 + (\gamma r^\rho \sin \rho\pi)^2} \leq \frac{r}{(\gamma r^\rho \sin \rho\pi)^2} \left|\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right| \leq \frac{Cr^{2-2\rho}}{(\gamma r^\rho \sin \rho\pi)^2},$$

where $C = -\lambda_1^{-1} + \gamma r_0^{\rho-1} + r_0^{-1}$. Hence,

$$\begin{aligned} \int_{r_0}^\infty e^{-\tau_0 r} a_\rho(\lambda, r) \frac{r \left|\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right|}{\left(\frac{-r}{\lambda} + \gamma r^\rho \cos \rho\pi + 1\right)^2 + (\gamma r^\rho \sin \rho\pi)^2} dr &\leq \int_{r_0}^\infty e^{-\tau_0 r} \frac{1}{\gamma\pi} \frac{1}{r^{\rho+1} \sin \rho\pi} \frac{Cr^{2-2\rho}}{(\gamma r^\rho \sin \rho\pi)^2} dr \\ &= \frac{C}{\pi(\gamma \sin \rho\pi)^3} \int_{r_0}^\infty e^{-\tau_0 r} r^{1-3\rho} dr \leq \frac{C}{\pi(\gamma r_0^\rho \sin \rho\pi)^3} \int_{r_0}^\infty e^{-\tau_0 r} r dr \\ &\text{(change of variables: } \xi = \tau_0 r, d\xi = \tau_0 dr) \\ &= \frac{C}{\pi(\gamma \sin \rho\pi)^3} \int_0^\infty e^{-\xi} \frac{\xi}{\tau_0} \frac{d\xi}{\tau_0} = \frac{C}{\pi(\gamma \sin \rho\pi)^3} \frac{\Gamma(2)}{\tau_0^2} = \frac{C_2}{\tau_0^2}. \end{aligned} \quad (4)$$

From the estimate (3), (4) it follows that for all $\lambda \geq \lambda_1 > 0$,

$$D_\lambda A_\rho(\lambda, \tau_0) \leq -\frac{2}{\lambda^2} \left(\frac{C_1}{\tau_0^{\rho+1}} + \frac{C_2}{\tau_0^2} \right) < 0.$$

Non-local problem

To solve the non-local problem (1), we take $u(t) = V(t) + W(t)$ and solve the following problems to try to solve this problem:

$$\begin{cases} D_t V(t) + A(1 + \gamma D_t^\rho) V(t) = f(t), & 0 < t \leq T; \\ V(0) = 0, \end{cases} \quad (5)$$

$$\begin{cases} D_t W(t) + A(1 + \gamma D_t^\rho) W(t) = 0, & 0 < t \leq T; \\ W(\tau_0) = \alpha W(0) + \psi, & 0 < \tau_0 \leq T, \end{cases} \quad (6)$$

where a function $\psi \in H$ is given, τ_0 is any fixed number from $(0, T]$ and $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$ is a fixed real number.

It is easy to verify that the function $u(t) = V(t) + W(t)$ is a solution of the problem (6) if $\psi = \varphi - V(\tau_0)$, $V(t)$ and $W(t)$ are solutions of the problems (5) and (6), respectively. As a result, solving the auxiliary problems (5) and (6) suffices to solve the main problem (1).

The auxiliary problem (5) is solved in [15]. Let us formulate the corresponding result:

Theorem 1. *Let $\varepsilon \in (0, 1)$ and $f(t) \in C([0, T]; D(A^\varepsilon))$. Then the Cauchy problem (5) has a unique solution and takes the following form:*

$$V(t) = \sum_{k=1}^{\infty} \left[\int_0^t B_{\rho}(\lambda_k, t - \tau) f_k(\tau) d\tau \right] v_k.$$

Furthermore, there exists a positive number C_ε such that the following coercive inequality is satisfied:

$$\|D_t V(t)\|^2 + \|AV(t)\|^2 + \|AD_t^\rho V(t)\|^2 \leq C_\varepsilon \max_{t \in [0, T]} \|f(t)\|_\varepsilon^2, \quad 0 \leq t \leq T. \quad (7)$$

Let us consider the non-local problem (6). We present the following theorem.

Theorem 2. *Let $\psi \in D(A)$.*

If $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$, then the problem (6) has a unique solution and this solution has the form

$$W(t) = \sum_{k=1}^{\infty} \frac{\psi_k}{A_{\rho}(\lambda_k, \tau_0) - \alpha} A_{\rho}(\lambda_k, t) v_k. \quad (8)$$

Moreover, the following coercive type inequality applies for some constant $C_\varepsilon > 0$:

$$\|D_t W(t)\|^2 + \|AW(t)\|^2 + \|AD_t^\rho W(t)\|^2 \leq C_\alpha \|\psi\|_1^2, \quad 0 < t \leq T. \quad (9)$$

Proof. Let $W(t)$ be a solution to the non-local problem (6). By the completeness of the system $\{v_k\}$ in H , the solution can be represented as:

$$W(t) = \sum_{k=1}^{\infty} T_k(t) v_k.$$

By substituting this expression into equation (6), we obtain the non-local problem:

$$\begin{cases} D_t T_k(t) + \lambda_k T_k(t) + \gamma \lambda_k D_t^\rho T_k(t) = 0; \\ T_k(\tau_0) = \alpha T_k(0) + \psi_k, \end{cases} \quad (10)$$

where $k \geq 1$, $\tau_0 \in (0, T]$ is the fixed point and ψ_k is the Fourier coefficient of the element $\psi \in H$.

Denote $b_k = T_k(0)$, $k = 1, 2, \dots$. Then according to Lemma 1 the unique solution to problem (10) has the form

$$T_k(t) = b_k A_{\rho}(\lambda_k, t).$$

To find the unknown numbers b_k , we use the non-local conditions of (10)

$$b_k A_{\rho}(\lambda_k, \tau_0) = \alpha b_k + \psi_k.$$

Then, we have the solution

$$T_k(t) = \frac{\psi_k}{A_{\rho}(\lambda_k, \tau_0) - \alpha} A_{\rho}(\lambda_k, t).$$

Since $A_{\rho}(\lambda_k, \tau_0) \neq \alpha$ (by Lemma 2), we obtain the formal solution (10) to the problem (6).

Let us prove the uniqueness of the solution to problem (10). The solution to problem (10) with the condition ψ_k has been defined subject to the condition $T_k(\tau_0) = T_k(0)$. According to Lemma 2, this implies that $b_k \equiv 0$, and consequently $T_k(t) \equiv 0$. By the completeness of the system $\{v_k\}$, it follows that $W(t) \equiv 0$.

Let us obtain an estimate for $W(t)$. Let $S_m(t)$ be the partial sums of the series (8)

$$S_m(t) = \sum_{k=1}^m \frac{\psi_k}{A_{\rho}(\lambda_k, \tau_0) - \alpha} A_{\rho}(\lambda_k, t) v_k.$$

Consider the action of the operator A to the partial sum $S_m(t)$, we obtain the following

$$AS_m(t) = \sum_{k=1}^m \frac{\psi_k}{A_{\rho}(\lambda_k, \tau_0) - \alpha} A_{\rho}(\lambda_k, t) \lambda_k v_k.$$

Applying Parseval's equality, Lemma 2, to get

$$\|AS_m(t)\|^2 = \sum_{k=1}^m \lambda_k^2 \left| \frac{\psi_k}{A_\rho(\lambda_k, \tau_0) - \alpha} A_\rho(\lambda_k, t) \right|^2 \leq C_\alpha \sum_{k=1}^m \lambda_k^2 |\psi_k|^2 \leq C_\alpha \|\psi\|_1^2,$$

where C_α is a constant positive number. This implies that if $\psi \in D(A)$, then $AW(t) \in C([0, T], D(A))$.

Let us obtain an estimate of $D_t S_m(t)$. Applying Lemma 5

$$D_t S_m(t) = \sum_{k=1}^m \frac{\psi_k}{A_\rho(\lambda_k, \tau_0) - \alpha} D_t A_\rho(\lambda_k, t) v_k = - \sum_{k=1}^m \frac{\psi_k}{A_\rho(\lambda_k, \tau_0) - \alpha} \lambda_k B_\rho(\lambda_k, t) v_k.$$

Applying Parseval's equality, Lemma 4, to get

$$\|D_t S_m(t)\|^2 = \left| \sum_{k=1}^m \frac{\psi_k}{A_\rho(\lambda_k, \tau_0) - \alpha} \lambda_k B_\rho(\lambda_k, t) \right|^2 \leq C_\alpha \sum_{k=1}^m |\psi_k|^2 |\lambda_k B_\rho(\lambda_k, t)|^2 \leq C_\alpha \sum_{k=1}^m |\psi_k|^2 t^{2(\rho-1)} \leq C_\alpha \|\psi\|_1^2,$$

where C_α is constant positive number.

Now we evaluate $A(D_t S_m(t))$. To do this, we use the basic equation we are studying

$$A(D_t S_m(t)) = -\frac{1}{\gamma} (D_t S_m(t) + AS_m(t)).$$

Then $\|A(D_t S_m(t))\|^2 \leq C_\alpha \|\psi\|_1^2$. Theorem 2 is completely proved.

For the non-local problem (1), we derive the following theorem by combining the statements of the previous two theorems Theorem 1 and Theorem 2.

Theorem 3. Let $f(t) \in C([0, T]; D(A^\varepsilon))$, and $\varphi \in D(A)$.

If $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$, then the problem (1) has a unique solution and this solution has the form

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{A_\rho(\lambda_k, \tau_0) - \alpha} A_\rho(\lambda_k, t) + V_k(t) \right] v_k, \quad (11)$$

where

$$V_k(t) = \int_0^t B_\rho(\lambda_k, t - \tau) f_k(\tau) d\tau.$$

Moreover, there are constants $C_\varepsilon > 0$, $C_\alpha > 0$ such that the following coercive type inequality holds:

$$\|D_t u(t)\|^2 + \|Au(t)\|^2 + \|AD_t^\rho u(t)\|^2 \leq C_\varepsilon \max_{t \in [0, T]} \|f(t)\|_\varepsilon^2 + C_\alpha \|\varphi\|_1^2, \quad 0 < t \leq T. \quad (12)$$

Proof. Let $f(t) \in C([0, T]; D(A^\varepsilon))$ for some $\varepsilon \in (0, 1)$, and let $\varphi \in D(A)$. As noted above, if we define $\psi = \varphi - V(\tau_0) \in D(A)$, where $V(t)$ and $W(t)$ are the solutions of problems (5) and (6), respectively, then the function $u(t) = V(t) + W(t)$ is a solution of the problem considered (1).

If $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$, then the existence and uniqueness of solutions to problems (5) and (6) implies the existence and uniqueness of the problem (1). We write the solution in the form (11).

Furthermore, the estimate (12) follows directly from the estimates (7) and (9). Theorem 3 is completely proved.

Inverse problem

Let us consider the inverse problem of finding the following source function:

$$\begin{cases} D_t u(t) + A(1 + \gamma D_t^\rho) u(t) = f, & \gamma > 0, \quad 0 < t \leq T; \\ u(\tau_0) = \alpha u(0) + \varphi, & 0 < \tau_0 \leq T, \end{cases} \quad (13)$$

with the addition condition

$$u(\tau) = \Psi, \quad 0 < \tau \leq T, \quad \tau \neq \tau_0, \quad (14)$$

where τ is a fixed point, $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$ is an arbitrary constant given, and the unknown element $f \in H$, does not depend on t and the given elements $\Psi, \varphi \in H$.

Let us give the definition of the inverse problem.

Definition 2. A pair $\{u(t), f\}$ of functions $u(t) \in C([0, T]; H)$ and $f \in H$ with properties $Au(t), D_t u(t), AD_t^\rho u(t) \in C([0, T]; H)$ and satisfying conditions (13), (14) is called the solution of the inverse problem (13), (14).

Before formulating the main result, consider the following quantity

$$\Delta(\tau, \tau_0, \lambda_k) = \alpha(1 - A_\rho(\lambda_k, \tau)) + A_\rho(\lambda_k, \tau) - A_\rho(\lambda_k, \tau_0). \quad (15)$$

Lemma 7. Let $0 < \tau < \tau_0 < T$. Then there is a constant c , depending on τ, τ_0 and λ_k , such that for any $k \geq 1$ one has

$$|\Delta(\tau, \tau_0, \lambda_k)| \geq c.$$

Proof. If $0 < \tau < \tau_0 < T$, then according Lemma 2, $A_\rho(\lambda_k, t)$ decreases, $\Delta(\tau, \tau_0, \lambda_k) \neq 0$ and the modulus of $\Delta(\tau, \tau_0, \lambda_k)$ is greater than the number c .

Now we state the main theorem of this section.

Theorem 3. Let $\varphi, \Psi \in D(A)$ and $\alpha \notin [C(\rho, \gamma, \lambda_1), 1) \cup \{0\}$. Then the inverse problem (13), (14) has a unique solution $\{u(t), f\}$ and this solution has the following form

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{A_\rho(\lambda_k, \tau_0) - \alpha} A_\rho(\lambda_k, t) - V_k(t) \right] v_k, \quad (16)$$

where

$$V_k(t) = \int_0^t B_\rho(\lambda_k, t - \tau) f_k d\tau = \frac{f_k}{\lambda_k} (1 - A_\rho(\lambda_k, t)),$$

and

$$f = \sum_{k=1}^{\infty} \left[\frac{\varphi_k A_\rho(\lambda_k, \tau)}{\Delta(\tau, \tau_0, \lambda_k)} + \frac{\Psi_k (\alpha - A_\rho(\lambda_k, \tau_0))}{\Delta(\tau, \tau_0, \lambda_k)} \right] \lambda_k v_k. \quad (17)$$

Proof. If f is known, the unique solution to problem (13) has the form (1). Since f is not dependent on t , it is simple to verify that the formal solution is a problem (16).

Using the additional condition (14), we obtain

$$\frac{\varphi_k - V_k(\tau_0)}{A_\rho(\lambda_k, \tau_0) - \alpha} A_\rho(\lambda_k, \tau) - V_k(\tau) = \Psi_k.$$

Simplifying by substituting $V_k(\tau), V_k(\tau_0)$, to get

$$f_k = \frac{\varphi_k A_\rho(\lambda_k, \tau) + \Psi_k (\alpha - A_\rho(\lambda_k, \tau_0))}{\alpha(1 - A_\rho(\lambda_k, \tau)) + A_\rho(\lambda_k, \tau) - A_\rho(\lambda_k, \tau_0)} \lambda_k.$$

We have

$$f_k = \left[\frac{\varphi_k A_\rho(\lambda_k, \tau)}{\Delta(\tau, \tau_0, \lambda_k)} + \frac{\Psi_k (\alpha - A_\rho(\lambda_k, \tau_0))}{\Delta(\tau, \tau_0, \lambda_k)} \right] \lambda_k \quad (18)$$

$$\equiv f_{k,1} + f_{k,2}.$$

We obtain the formal series (18) for the unknown function using their Fourier coefficients $f : f = \sum_{k=1}^{\infty} (f_{k,1} + f_{k,2}) v_k$. In the case, we will show the convergence of the series (18). Given F_n , which represent the partial sums of the (18), we may write

$$f : f = \sum_{k=1}^n [f_{k,1} + f_{k,2}]^2 \leq 2 \sum_{k=1}^n f_{k,1}^2 + 2 \sum_{k=1}^n f_{k,2}^2 \equiv 2F_{1,n} + 2F_{2,n}.$$

According to Lemma 2 and Lemma 7, then

$$F_{1,n} \leq \sum_{k=1}^n \left| \frac{\varphi_k A_\rho(\lambda_k, \tau)}{\Delta(\tau, \tau_0, \lambda_k)} \right|^2 \lambda_k^2 \leq C \|\varphi\|_1^2.$$

Again according to Lemma 2 and Lemma 7, to get

$$F_{2,n} \leq \sum_{k=1}^n \left| \frac{\Psi_k (\alpha - A_\rho(\lambda_k, \tau_0))}{\Delta(\tau, \tau_0, \lambda_k)} \right|^2 \lambda_k^2 \leq C \|\Psi\|_1^2.$$

Therefore, if $\varphi, \Psi \in D(A)$, then $f \in H$ can be obtained from estimates of $F_{1,n}, F_{2,n}$.

Uniqueness. Assume that we have $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$ as our two solutions. It is necessary to demonstrate that $u(t) \equiv u_1(t) - u_2(t) = 0$ and $f \equiv f_1 - f_2 = 0$. Given that the issue is linear, we must solve the following to find $u(t)$ and f :

$$D_t u(t) + A(1 + \gamma D_t^\rho) u(t) = f, \quad 0 < t \leq T; \quad (19)$$

$$u(\tau_0) = \alpha u(0), \quad 0 < \tau_0 \leq T; \quad (20)$$

$$u(\tau) = 0, \quad \tau \neq \tau_0, \quad (21)$$

where τ_0 - fixed point.

Assume that this issue has a solution $u(t)$ and that $u_k(t) = (u(t), v_k)$. Then, by virtue of the self-adjointness of operator A and equation (19)

$$\begin{aligned} D_t u(t) = (D_t u(t), v_k) &= -(A(1 + \gamma D_t^\rho) u(t), v_k) + (f, v_k) = -(u(t) + \gamma D_t^\rho u(t), A v_k) + (f, v_k) \\ &= -\lambda_k (u(t) + \gamma D_t^\rho u(t), v_k) + f_k = -\lambda_k (u_k(t) + \gamma D_t^\rho u_k(t)) + f_k, \quad 0 < t < T. \end{aligned}$$

Therefore, considering (20) and (21), we have the following problem

$$D_t u_k(t) + \gamma \lambda_k D_t^\rho u_k(t) + \lambda_k u_k(t) = f_k, \quad 0 < t < T; \quad u_k(\tau_0) = \alpha u_k(0), \quad u_k(\tau) = 0.$$

If f_k is known, and use the non-local condition to get it (11), we have the following

$$u_k(t) = \frac{V_k(\tau_0)}{\alpha - A_\rho(\lambda_k, \tau_0)} A_\rho(\lambda_k, t) + V_k(t),$$

where

$$V_k(t) = \int_0^t B_\rho(\lambda_k, t - \tau) f_k d\tau = \frac{f_k}{\lambda_k} (1 - A_\rho(\lambda_k, t)).$$

Use $u_k(\tau) = 0$ to get

$$u_k(\tau) = \frac{V_k(\tau_0)}{\alpha - A_\rho(\lambda_k, \tau_0)} A_\rho(\lambda_k, \tau) + V_k(\tau) = 0.$$

Simplifying by substituting $V_k(\tau)$, $V_k(\tau_0)$, we obtain

$$\frac{f_k}{\lambda_k} [\alpha(1 - A_\rho(\lambda_k, \tau)) + A_\rho(\lambda_k, \tau) - A_\rho(\lambda_k, \tau_0)] = 0.$$

And from this

$$\frac{f_k}{\lambda_k} \Delta(\tau, \tau_0, \lambda_k) = 0.$$

According to Lemma 7, $f_k = 0$ for all k . Due to the complete system of eigenfunctions $\{v_k\}$, we get $f = 0$ and $u(t) \equiv 0$, as needed. The proof of Theorem 3 is complete.

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METHODS OF ASSEMBLING OF THE DEGREE SERIES OF SEVERAL VARIABLES OF THE COMPLEX NUMBERS

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Abstract. Here we shall describe some of the simplest examples of domains in the space $\mathbb{C}^n$. As usual, a domain is an open connected set, where openness means that along with any point of it the set also contains a neighborhood of that point, and connectedness of an open set D means that, for any points $z', z'' \in D$ there exists a continuous arc $\gamma: [0, 1] \rightarrow D$ for which $\gamma(0) = z'$ and $\gamma(1) = z''$.

Key words: degree series, several variables, complex numbers, point, function.

МЕТОДЫ ПОСТРОЕНИЯ СТЕПЕННЫХ РЯДОВ НЕСКОЛЬКИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ

Аннотация. Здесь описываются некоторые из простейших примеров областей в пространстве $\mathbb{C}^n$. Как обычно, область — это открытое связное множество, где открытость означает, что вместе с любой точкой множество содержит также её окрестность. Связность открытого множества D означает, что для любых точек $z', z'' \in D$, $z' \in D, z'' \in D$ существует непрерывная кривая $\gamma: [0, 1] \rightarrow D$ такая что $\gamma(0) = z'$, $\gamma(1) = z''$.

Ключевые слова: степенные ряды, несколько переменных, комплексные числа, точка, функция.

BIR NECHTA KOMPLEKS O'ZGARUVCHILARNING DARAJALI QATORLARINI TUZISH USULLARI

Annotatsiya. Bu yerda $\mathbb{C}^n$ fazosidagi eng sodda sohalarining ayrim misollari bayon etiladi. Odatdagidek, soha — bu ochiq va bog'langan to'plam bo'lib, ochiqlik har qanday nuqta bilan birga uning atrofidagi yaqinlik (atrof) ham to'plamga tegishli ekanligini anglatadi. Ochiq D to'plamining bog'langanligi shuni bildiradiki, har qanday $z', z'' \in D$, $z' \in D, z'' \in D$ nuqtalar uchun $\gamma: [0, 1] \rightarrow D$ mavjud bo'lib, $\gamma(0) = z'$, $\gamma(1) = z''$.

Kalit so'zlar: darajali qatorlar, bir nechta o'zgaruvchilar, kompleks sonlar, nuqta, funksiya.

In this section, we introduce double series and we shall give the definition of their convergence and divergence. Then we study relationship between double and iterated series, and we give a sufficient condition for equality of iterated series.

Definition. A function $l: \mathbb{C}^n \rightarrow \mathbb{C}$ is said to be R -linear (respectively C -linear) if:

- (a) $l(z' + z'') = l(z') + l(z'')$ for all $z', z'' \in \mathbb{C}^n$,
- (b) $l(\lambda z) = \lambda l(z)$ for all $z \in \mathbb{C}^n$ and all $\lambda \in R$ (respectively all $\lambda \in \mathbb{C}$).

Any R -linear function in $\mathbb{C}^n$ has the form

$$l(z) = \sum_{v=1}^n (a_v z_v + b_v \bar{z}_v), \quad a_v, b_v \in \mathbb{C},$$

and any C -linear function has the form

$$l(z) = \sum_{v=1}^n a_v z_v, \quad a_v \in \mathbb{C}$$

An R -linear function l is C -linear if and only if

$$l(iz) = il(z).$$

Definition. A function $f: U \rightarrow \mathbb{C}$, where U is a neighborhood of a point $z \in \mathbb{R}^n$, is said to be $\mathbb{R}$ -differentiable (respectively, $\mathbb{C}$ -differentiable) at z if

$$f(z+h) = f(z) + l(h) + o(h),$$

where l is some $\mathbb{R}$ -linear (respectively $\mathbb{C}$ -linear) function, and $o(h)/|h| \rightarrow 0$ as $h \rightarrow 0$.

The function l is called the differential of f at z and is denoted by the symbol df . Setting

$$h = dz = dx + i dy,$$

where

$$dz = (dz_1, \dots, dz_n)$$

is a complex vector and

$$dx = (dx_1, \dots, dx_n) \quad \text{and} \quad dy = (dy_1, \dots, dy_n)$$

are real vectors, we can, in the general case of $\mathbb{R}$ -differentiability, write the differential in the form

$$df = \sum_{v=1}^n \left(\frac{\partial f}{\partial x_v} dx_v + \frac{\partial f}{\partial y_v} dy_v \right)$$

or, after passing to complex coordinates, in the form

$$df = \sum_{v=1}^n \left(\frac{\partial f}{\partial z_v} dz_v + \frac{\partial f}{\partial \bar{z}_v} d\bar{z}_v \right) \quad (1)$$

where we have introduced the notations

$$\frac{\partial f}{\partial z_v} = \frac{1}{2} \left(\frac{\partial f}{\partial x_v} - i \frac{\partial f}{\partial y_v} \right),$$

$$\frac{\partial f}{\partial \bar{z}_v} = \frac{1}{2} \left(\frac{\partial f}{\partial x_v} + i \frac{\partial f}{\partial y_v} \right),$$

$$v = 1, 2, \dots, n.$$

The first sum in (1) is denoted by the symbol ∂f and the second by the symbol $\bar{\partial} f$, so that

$$\partial = \sum_{v=1}^n \frac{\partial}{\partial z_v} dz_v, \quad \bar{\partial} = \sum_{v=1}^n \frac{\partial}{\partial \bar{z}_v} d\bar{z}_v, \quad v = 1, 2, \dots, n.$$

Theorem. For a function f that is $\mathbb{R}$ -differentiable at a point $z \in \mathbb{C}^n$ to be $\mathbb{C}$ -differentiable at z it is necessary and sufficient that the Cauchy-Riemann conditions

$$\bar{\partial} f = 0$$

hold.

We see from (i) that

$$df(ih) = i\partial f(h) - i\bar{\partial} f(h)$$

and

$$idf(h) = i\partial f(h) + i\bar{\partial} f(h).$$

Therefore the condition of $\mathbb{C}$ -differentiability

$$df(ih) = idf(h)$$

is equivalent to the condition

$$\bar{\partial} f(h) = 0 \quad \text{for all } h \in \mathbb{C}^n.$$

The Cauchy-Riemann conditions

$$\bar{\partial} f = 0$$

are equivalent to a system of n complex equations

$$\frac{\partial f}{\partial \bar{z}_v} = \frac{1}{2} \left(\frac{\partial f}{\partial x_v} + i \frac{\partial f}{\partial y_v} \right) = 0,$$

or the equivalent system of $2n$ real equations

$$\frac{\partial u}{\partial x_v} = \frac{\partial v}{\partial y_v}, \quad \frac{\partial u}{\partial y_v} = -\frac{\partial v}{\partial x_v}, \quad v = 1, 2, \dots, n.$$

Where

$$u = \operatorname{Re} f \quad \text{and} \quad v = \operatorname{Im} f.$$

For $n > 1$ this system is overdetermined (contains $2n$ equations relative to two unknown functions) and this circumstance is a principal difference of multidimensional complex analysis from the one-dimensional theory.

Definition. A function f is said to be holomorphic at a point $z \in \mathbb{C}^n$ if it is $\mathbb{C}$ -differentiable in some neighborhood of this point. On an open set the concepts of $\mathbb{C}$ -differentiability and holomorphy are the same.

We remark that in the definition of holomorphy on an arbitrary (not necessarily open) set M there is a subtlety, which is seen from the following example.

Example. Suppose the set $M \subset \mathbb{C}^2$ consists of two closed balls

$$\overline{B}_1 = \{|x - (0,1)| \leq \frac{1}{2}\} \quad \text{and} \quad \overline{B}_2 = \{|x + (0,1)| \leq \frac{1}{2}\},$$

joined by the line segment

$$L = \{x_1 = 0, x_2 = x_1, |x_2| \leq \frac{1}{2}\}.$$

On M we define the function

$$\begin{cases} z_1, & z \in \overline{B}_1 \\ 0, & z \in L \\ -z_1, & z \in \overline{B}_2. \end{cases}$$

It is obviously continuous on M , and for each point $z_0 \in M$ one can contract a neighborhood U_{z_0} into which f extends as a holomorphic function. In fact, for any point of $\overline{B}_1$, including the point

$$\left(0, \frac{1}{2}\right)$$

of intersection of $\overline{B}_1$ and L , we can take as such neighborhoods balls that do not intersect $\overline{B}_2$, and extend f into them by setting it equal to z_1 . For points of $\overline{B}_2$ we make an analogous construction, only we set $f(z) = -z_1$. Finally, for interior points of L we take balls that do not contain the ends of this line segment and set $f = 0$ in them. However, from the uniqueness theorem that we shall prove in subsection 5, it follows that f cannot be extended to a holomorphic function in any connected neighborhood Ω of the whole set M . In fact, it follows from this theorem that there does not exist a function that is holomorphic in Ω that is equal to z_1 in one ball from Ω and to $-z_1$ in the other.

From this example we see that it is necessary to distinguish functions that are locally holomorphic on a set, i.e., functions that can be extended locally to a holomorphic function at each point of the set, and globally holomorphic functions, which extend to functions that are holomorphic in a neighborhood of the whole set. Later on, as a rule, in speaking about the holomorphy of a function on a set, we shall have global holomorphy in mind.

The sum and product of functions that are holomorphic at a point $z \in \mathbb{C}^n$ are also holomorphic at this point, so that the set of all functions holomorphic at the point z forms a ring, which is denoted by the symbol Π_z . The ring of functions that are holomorphic in a domain $D \subset \mathbb{C}^n$ is denoted by the symbol $\Pi(D)$. From the Cauchy-Riemann conditions we see that a function f that is holomorphic in a neighborhood $U \subset \mathbb{C}^n$ of the point z^0 is holomorphic with respect to each variable z_v for fixed remaining variables (in a neighborhood of the point z_v^0 on the complex line

$$\{z: z_\mu = z_\mu^0, \mu \neq v\}.$$

Further, since the system splits into equations in each of which the derivative with respect to only one of the variables occurs, it seems that the condition of holomorphy with respect to a set of variables does not impose any constraints on the dependence on the remaining variables. However, some constraint is nevertheless established: it turns out that a function that is holomorphic in U with respect to each variable z_v separately is certainly $\mathbb{R}$ -differentiable with respect to all the variables jointly and then by Theorem 1 is holomorphic in U .

Pluriharmonic functions. We start with a simple remark: if the function

$$f = u + iv$$

is holomorphic at a point $z \in \mathbb{C}^n$, then the function

$$\bar{f} = u - iv$$

is $\mathbb{R}$ -differentiable in a neighborhood of this point and, for any $v = 1, 2, \dots, n$,

$$\frac{\partial \bar{f}}{\partial z_v} = \frac{1}{2} \left(\frac{\partial \bar{f}}{\partial x_v} - i \frac{\partial \bar{f}}{\partial y_v} \right) = \overline{\frac{\partial f}{\partial z_v}} = 0$$

there. Such functions $\bar{f}$ are termed antiholomorphic at the point z .

Suppose f is holomorphic at a point $z \in \mathbb{C}^n$. Then by the preceding remark we have

$$\partial u / \partial z_v = \frac{1}{2} \partial f / \partial z_v$$

for its real part

$$u = \frac{1}{2} (f + \bar{f})$$

in a neighborhood of z . We will also use the fact that the partial derivatives of a holomorphic function are also holomorphic. It follows from this that, for any

$$\mu, v = 1, 2, \dots, n,$$

$$\frac{\partial^2 u}{\partial z_\mu \partial z_v} = \frac{\partial}{\partial z_v} \left(\frac{\partial u}{\partial z_\mu} \right) = 0. \quad (2)$$

Separating the real and imaginary parts of the operator in the left-hand side of:

$$\frac{\partial}{\partial z_\mu} \frac{\partial}{\partial z_v} = \frac{1}{4} \left(\frac{\partial^2}{\partial x_\mu \partial x_v} + \frac{\partial^2}{\partial y_\mu \partial y_v} \right) + \frac{i}{4} \left(\frac{\partial^2}{\partial x_\mu \partial y_v} - \frac{\partial^2}{\partial y_\mu \partial x_v} \right),$$

we find that condition (2) splits into n^2 second-order partial equations:

$$\frac{\partial^2 u}{\partial x_\mu \partial x_v} + \frac{\partial^2 u}{\partial y_\mu \partial y_v} = 0, \quad \frac{\partial^2 u}{\partial x_\mu \partial y_v} - \frac{\partial^2 u}{\partial y_\mu \partial x_v} = 0 \quad (3)$$

$$(\mu, v = 1, 2, \dots, n).$$

If we use the operators ∂ and $\bar{\partial}$ introduced in (2) of subsection (3), then the system (2) can be rewritten as a single condition

$$\bar{\partial} \partial u = 0. \quad (4)$$

Definition. A function u of class C^2 in a domain $D \subset \mathbb{C}^n$ that satisfies the condition (4) at each point of D is said to be *pluriharmonic* in D .

Pluriharmonic functions are connected with holomorphic functions of several variables in the same way that harmonic functions (in $\mathbb{R}^2$) are connected with holomorphic functions of one variable. Namely, we have the following two theorems:

Theorem. The real and imaginary parts of a function f that are holomorphic in a domain $D \subset \mathbb{C}^n$ are pluriharmonic in D .

Proof. For the real part $u = \operatorname{Re} f$ the theorem has already been proved. Since if $f \in \zeta(D)$, so is $-if$, and since

$$\operatorname{Im} f = \operatorname{Re}(-if),$$

the theorem also holds for the imaginary part of f . The converse is also true, but generally only locally.

Theorem. For any function u that is pluriharmonic in a neighborhood U of the point

$$(x^0, y^0) \in \mathbb{R}^{2n},$$

there exists a function f , holomorphic at

$$z^0 = x^0 + iy^0,$$

whose real (of imaginary) part is equal to u .

Proof. Instead of

$$du = \sum \left(\frac{\partial u}{\partial x_v} dx_v + \frac{\partial u}{\partial y_v} dy_v \right)$$

we consider the so-called conjugate differential

$$*du = \sum_{v=1}^n \left(-\frac{\partial u}{\partial y_v} dx_v + \frac{\partial u}{\partial x_v} dy_v \right).$$

The form first $w = *du$ is closed in U , since, because u is pluriharmonic, its coefficients are of class C^1 and

$$dw = \sum_{\mu, v=1}^n \left(\frac{\partial^2 u}{\partial x_v \partial y_\mu} - \frac{\partial^2 u}{\partial x_\mu \partial y_v} \right) (dx_\mu \wedge dx_v + dy_\mu \wedge dy_v) + \sum_{\mu, v=1}^n \left(\frac{\partial^2 u}{\partial x_\mu \partial x_v} + \frac{\partial^2 u}{\partial y_\mu \partial y_v} \right) dx_\mu \wedge dy_v = 0.$$

But every closed form is locally exact, so that in a neighborhood of z^0 there is a function v such that $*du = dv$: it is expressed by the integral

$$v(z) = \int_{z^0}^z *du,$$

which does not depend on the path since $*du$ is closed. But then in a neighborhood of z^0

$$\frac{\partial v}{\partial x_v} = -\frac{\partial u}{\partial y_v}, \quad \frac{\partial v}{\partial y_v} = \frac{\partial u}{\partial x_v},$$

and consequently the function

$$f = u + iv$$

is of class C^2 and satisfies the Cauchy-Riemann equations of the preceding subsection; by Theorem 1 of subsection it is holomorphic at z^0 and

$$u = \operatorname{Re} f.$$

Adding these equations for

$$v = 1, 2, \dots, n,$$

we will find that the Laplacian of the function u with respect to the variables

$$x_1, y_1, \dots, x_n, y_n$$

satisfies

$$\Delta u = \sum_{v=1}^n \left(\frac{\partial^2 u}{\partial x_v^2} + \frac{\partial^2 u}{\partial y_v^2} \right) = 0.$$

Consequently, *pluriharmonic functions form a subclass of the class of harmonic functions* in the space R^{2n} (obviously proper for $n > 1$). There naturally arises the question of the determination of a pluriharmonic function in a domain $D \subset R^{2n}$ from given boundary values (Dirichlet problem). This question is not solved as simply as in the case of harmonic functions. We illustrate the difficulties that arise using the example of one of the simplest domains, the polydisc

$$U = \{z \in \mathbb{C}^n : |z_v| < 1\}.$$

Since, a pluriharmonic function is harmonic with respect to each variable

$$z_v = x_v + iy_v$$

in the disc

$$\{|z_v| < 1\},$$

we can successively apply the Poisson integral from subsection 2 of the Appendix to Part 1. For any $z \in U$ we obtain

$$u(z) = \int_0^{2\pi} p(\zeta_1, z_1) dt_1 \dots \int_0^{2\pi} u(\zeta) p(\zeta_n, z_n) dt_n$$

where

$$\zeta_v = e^{it_v}, \quad \zeta = (\zeta_1, \dots, \zeta_n),$$

and

$$p_n(\zeta_v, z_v) = \frac{1}{2\pi} \frac{1 - |z_v|^2}{|\zeta_v - z_v|^2}$$

in the Poisson kernel. Denoting by

$$p_n(\zeta, z) = \prod_{v=1}^n p(\zeta_v, z_v)$$

the n -dimensional Poisson kernel, by

$$q_n = [0, 2\pi] \times \dots \times [0, 2\pi] \text{ (n times)}$$

the n -dimensional cube, and by

$$dt = dt_1 \dots dt_n$$

the volume element, we rewrite the Poisson integral formula in the following abbreviated form:

$$u(z) = \int u(\zeta) p_n(\zeta, z) dt.$$

The right-hand side of this formula includes only the values of u on the *skeleton* Γ of the polydisc, i.e., on the n -dimensional part of the boundary ∂U . From this it is clear that one cannot arbitrarily give the values of a pluriharmonic function u on the whole boundary of the polydisc. If in the right-hand side we substitute the values of some function $u(\zeta)$ that is continuous on Γ , then the function $u(z)$ defined in U by this formula will, as is not hard to check, satisfy the equations for all $v = 1, 2, \dots, n$.

Simplest properties of holomorphic functions. Here we establish a number of elementary properties of holomorphic functions of several variables that are analogous to properties of functions of one variable. For brevity we shall denote by

$$U = \{z \in \mathbb{C}^n : |z_v - a_v| < r_v, \quad v = 1, 2, \dots, n\}$$

the polydisc with center a and vector radius

$$r = (r_1, \dots, r_n).$$

We denote by

$$\varphi(u) \cap C(\bar{U})$$

the set of functions that are holomorphic in U and continuous in $\bar{U}$.

Theorem. Any function

$$f \in \varphi(u) \cap C(\bar{U})$$

at any point $z \in U$ is represented by a multiple Cauchy integral

$$f(z) = \frac{1}{(2\pi i)^n} \int \frac{f(\zeta) d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)},$$

where Γ is the skeleton of U , i.e., the product of the boundary circles

$$\gamma_v = \{|z_v - a_v| = r_v\}, \quad v = 1, 2, \dots, n.$$

Proof. Let z' and U' be the projections of z and U into $\mathbb{C}^{n-1}$; since for any $z' \in U'$ the function

$$f(z) = f(z', z_n)$$

is holomorphic in z_n in the disc

$$\{|z_v - a_v| < r_v\}$$

and is continuous in its closure, then by the Cauchy integral formula from theorem

$$f(z) = \frac{1}{2\pi i} \int \frac{f(z', \zeta_n) d\zeta_n}{\zeta_n - z_n}.$$

Remark. As we see from the proof of Theorem 1, in order for a function f to be representable by a multiple Cauchy integral it is sufficient for f to be holomorphic in each variable z_v in the disc $\{|z_v - a_v| < r_v\}$ and continuous with respect to the set of all variables in U . As in the case of one variable, from the Cauchy integral representation of a function we deduce the possibility of expanding it in a power series. For this we expand the kernel of the integral in a multiple geometric progression:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \times \frac{1}{\left(1 - \frac{z_1 - a_1}{\zeta_1 - a_1}\right) \dots \left(1 - \frac{z_n - a_n}{\zeta_n - a_n}\right)} = \frac{1}{\zeta - a} \sum_{k=1}^{\infty} \left(\frac{z - a}{\zeta - a}\right)^k,$$

where

$$k = (k_1, k_2, \dots, k_n)$$

is an integer vector,

$$|k| = k_1 + k_2 + \dots + k_n,$$

and

$$\left(\frac{z - a}{\zeta - a}\right)^k = \left(\frac{z_1 - a_1}{\zeta_1 - a_1}\right)^{k_1} \dots \left(\frac{z_n - a_n}{\zeta_n - a_n}\right)^{k_n}.$$

This expansion can be rewritten in the form

$$\frac{1}{\zeta - z} = \sum_{|k|=0}^{\infty} \frac{(z - a)^k}{(\zeta - a)^{|k|+1}},$$

where

$$k + 1 = (k_1 + 1, \dots, k_n + 1);$$

for any $z \in U$ it converges absolutely and uniformly in ζ on Γ . Multiplying it by the continuous (and hence bounded) function

$$f(\zeta)/(2\pi i)^n$$

on Γ and integrating term-by-term over Γ , we will obtain the desired assertion:

Theorem. If

$$f \in \varphi(u) \cap C(\overline{U}),$$

then at each point $z \in U$ it is represented as a multiple power series

$$f(z) = \sum_{|k|=0}^{\infty} c_k (z - a)^k$$

with coefficients

$$c_k = \frac{1}{(2\pi i)^n} \int \frac{f(\zeta) d\zeta}{(\zeta - a)^{k+1}}.$$

Remark. Any function $f \in \varphi(u)$ can be represented as the sum of a series at each point $z \in U$. For the proof it suffices to remark that the point z belongs to some polydisc

$$U' \subset\subset U$$

and to apply Theorem 2 to U' .

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A(z)-ANALITIK FUNKSIYALAR UCHUN EKSTREMUM PRINSIPI, KOSHI TIPIDAGI INTEGRAL TUSHUNCHASI VA ULARNING TATBIQLARI

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Annotatsiya. Ushbu maqolada $A(z)$ funksiyani antianalitik bo‘lgan holda $A(z)$ – analitik funksiyalarni qaraymiz. Bu maqolada $A(z)$ – analitik funksiyalar uchun ekstremum prinsipi va Koshi tipidagi integral tushunchasi kiritilgan, shuningdek, ularning tadbiqlariga doir bir qancha misollar ko‘rilgan.

Kalit so‘zlar: $A(z)$ – analitik funksiya, $A(z)$ – lemniskata, ekstremum prinsipi, Koshi tipidagi integral, funksiya maxsus nuqtasidagi chegirmasi.

ПРИНЦИП ЭКСТРЕМУМА, ИНТЕГРАЛ ТИПА КОШИ И ИХ ПРИЛОЖЕНИЯ ДЛЯ A(Z)-АНАЛИТИЧЕСКИХ ФУНКЦИЙ

Аннотация. Мы рассматриваем $A(z)$ – аналитические функции в случае, когда $A(z)$ антианалитическая функция. В данной статье введены принцип экстремума для $A(z)$ – аналитических функций и понятие интеграла типа Коши, а также рассмотрен ряд примеров, иллюстрирующих их применения.

Ключевые слова: $A(z)$ – аналитическая функция, $A(z)$ – лемниската, принцип экстремума, типа интеграл Коши, вычет в полярных точках.

THE EXTREMUM PRINCIPLE, CAUCHY-TYPE INTEGRAL, AND THEIR APPLICATIONS FOR A(Z)-ANALYTIC FUNCTIONS

Abstract. We consider $A(z)$ – analytic functions in case when $A(z)$ is antianalytic function. In this article, the extremum principle for $A(z)$ – analytic functions and the concept of Cauchy-type integrals are introduced, and a number of examples illustrating their applications are also considered.

Key words: $A(z)$ – analytic function, $A(z)$ – lemniscate, extremum principle, Cauchy-type integral, residue at poles.

Kirish. $A(z)$ -analitik funksiyalar to‘g‘risida dastlabki tushunchalar. Kvazikonform akslantirishlar nazariyasi bilan bog‘liq bo‘lgan ushbu

$$\frac{\partial f}{\partial \bar{z}} = A(z) \frac{\partial f}{\partial z} \quad (1)$$

Beltrami tenglamasining yechimini qaraylik. Bu yerda, $A(z)$ funksiya o‘lchovli va qaralayotgan $D \subset \square$ sohaning deyarli barcha nuqtalarida $|A(z)| \leq C$ shartni qanoatlantiradi, bu yerda $C < 1$. Adabiyotlarda (1) tenglamaning yechimi $A(z)$ -analitik funksiya deb ataladi [4].

Aytaylik, $A(z)$ antianalitik funksiya va $D \subset \square$ sohada $\forall z \in D$ uchun $|A(z)| \leq C < 1$ o‘rinli bo‘lsin. Shunday qilib, antianalitik funksiyalar cheksiz silliq bo‘lganligidan $O_A(D) \subset C^\infty(D)$ ekanligi kelib chiqadi [5].

Asosiy qism. 1-teorema (Koshi teoremasining analogi, [6]). Faraz qilaylik, $D \subset \square$ soha bo'lakli silliq chegaradan iborat bo'lsin. Agar $f \in O_A(D) \cap C(\bar{D})$ bo'lsa, u holda

$$\int_{\partial D} f(z)(dz + A(z)d\bar{z}) = 0$$

bo'ladi.

$A(z)$ – analitik funksiyalarni o'rganishda, $A(z)$ – antianalitik funksiya bo'lishida Koshi yadrosi katta ahamiyatga ega:

$$K(z, \zeta) = \frac{1}{2\pi i} \frac{1}{z - \zeta + \int_{\gamma(z, \zeta)} \overline{A(\tau)} d\tau}, \quad (2)$$

bu yerda, $\gamma(z, \zeta) - \zeta$, $z \in D$ nuqtalarni tutashtiruvchi egri chiziqli yoki ζ nuqtadan z nuqtaga boruvchi ixtiyoriy yo'lni $\gamma(z, \zeta)$ bilan belgilaymiz. Shunday qilib D soha bir bog'lamli va $\bar{A}(z)$ – holomorf funksiya bo'lsa, u holda $I(z) = \int_{\gamma(\zeta, z)} \bar{A}(\tau) d\tau$ integrallash yo'liga bog'liq bo'lmaydi, u

dastlabki holat $I'(z) = \bar{A}(z)$ bilan to'g'ri keladi [8].

Agar $D \subset \square$ soha bo'lsa, u holda quyidagi teorema o'rinli:

2-teorema [8]. $K(z, \zeta)$ yadro $z = \zeta$ dan tashqari boshqa nuqtalarda $A(z)$ – analitik funksiya bo'ladi, ya'ni $K \in O_A(D \setminus \zeta)$. Shuningdek, $K(z, \zeta)$ uchun $z = \zeta$ nuqta birinchi tartibli qutb nuqta bo'ladi.

1-tasdiq [8]. Agar D soha qavariq bo'lmasa, faqatgina bir bog'lamli bo'lsa, u holda $\psi(z, \zeta) = z - \zeta + \int_{\gamma(z, \zeta)} \overline{A(\tau)} d\tau$ funksiya ham D sohada bir qiymatli aniqlanadi, ammo u boshqa yakka nollarga $\zeta: \psi(z, \zeta) \neq 0, z \in P = \{\zeta, \zeta_1, \zeta_2, \zeta_3, \dots\}$ ega bo'lishi mumkin. Biroq $\psi \in O_A(D)$, $\psi(z, \zeta) \neq 0, z \notin P$ da va $K(z, \zeta)$ yadro $D \setminus P$ da oddiy qutb P to'plamining nuqtalarida $A(z)$ – analitik funksiya bo'ladi.

Aytaylik, $D \subset \square$ qavariq soha bo'lsin. $\psi(z, \xi) = z - \xi + \int_{\gamma(\xi, z)} \bar{A}(\tau) d\tau \in O_A(D)$ funksiya

ichki akslantirishni amalga oshiradi. Xususan, $L(a, r) = \{z \in D: |\psi(a, z)| < r\}$ to'plam D da ochiq to'plamni ifodalaydi, bu yerda $\gamma(a, z) - a, z \in D$ nuqtalarni tutashtiruvchi silliq chiziqli. Yetarlicha kichik $r > 0$ uchun bu to'plam D da kompakt yotadi va nuqtani o'z ichiga oladi. Bu to'plamni markazi a bo'lgan $A(z)$ -lemniskata deyiladi va $L(a, r)$ ko'rinishda belgilanadi [8].

3-teorema (Koshi formulasining analogi, [7]). Faraz qilaylik, $D \subset \square$ – qavariq soha va $G \subset D - \partial G$ chegarasi bo'lakli silliq soha bo'lsin. U holda ixtiyoriy $f \in O_A(D) \cap C(\bar{D})$ funksiya uchun quyidagi formula o'rinli

$$f(z) = \int_{\partial G} K(z, \xi) f(\xi) (d\xi + A(\xi) d\bar{\xi}), \quad z \in G. \quad (3)$$

$A(z)$ -analitik funksiyalar sinfida chegirmalar nazariyasi. Aytaylik, $f(z)$ funksiya $D \setminus \{a_1, a_2, \dots, a_n\}$ sohada $A(z)$ – analitik va ∂D da uzluksiz bo'lsin. Bu yerda $a_1, a_2, \dots, a_n$ – nuqtalar D sohaning ichida yotuvchi chekli yoki sanoqli yakka nollardan maxsus nuqtalar bo'lsin.

Shunday $r > 0$ mavjudki, $L(a_k, r) \cap L(a_\ell, r) = \emptyset, k \neq \ell$ bo'ladi.

Aytaylik, $G_r = z \in D: |z - \xi| > r, \forall \xi \in \partial D$ va $\bigcup_{k=1}^n L(a_k, r) \subset G_r$ bo'lsin. ∂G_r -ixtiyoriy bo'lakli silliq yopiq egri chiziq bo'lib, $a_1, a_2, \dots, a_n$ nuqtalarni o'z ichiga oladi va butunligicha D da yotadi. $f(z)$ funksiya $\partial G_r = \bigcup_{k=1}^n \partial L(a_k, r)$ yopiq chiziq bilan chegaralangan sohaning har bir nuqtasida $A(z)$ – analitik bo'lganligi uchun Koshi teoremasiga ko'ra quyidagi tenglik o'rinli:

$$\int_{\partial G_r} f(\xi) \omega(\xi) = \sum_{k=1}^n \int_{\partial L(a_k, r)} f(\xi) \omega(\xi)$$

bu yerda $\omega(z) = dz + A(z)d\bar{z}$.

1-ta'rif [9]. $A(z)$ – analitik $f(z)$ funksiyaning a nuqtadagi chegirmasi deb

$$\frac{1}{2\pi i} \int_{\partial L(a, r)} f(\xi) \omega(\xi)$$

ifodaning qiymatiga aytiladi va quyidagicha belgilanadi:

$$\operatorname{res}_{z=a} f(z) := \frac{1}{2\pi i} \int_{\partial L(a, r)} f(\xi) \omega(\xi). \quad (4)$$

4-teorema [9]. Faraz qilaylik, $f(z)$ funksiya $G \subset \subset D$ to'plamning maxsus nuqtalaridan boshqa nuqtalarida $A(z)$ – analitik funksiya bo'lsin va ∂G chegarasi maxsus nuqtalarni o'zida saqlamasin. U holda

$$\int_{\partial G} f(\xi) \omega(\xi) = 2\pi i \sum_{k=1}^n \operatorname{res}_{z=a_k} f(z)$$

tenglik o'rinli bo'ladi.

Masala [9]. Fiksirlangan $\xi \in D$ nuqtada quyidagi yadroni qaraymiz:

$$K_n(\xi, z) = \frac{n!}{2\pi i} \cdot \frac{1}{\psi(\xi, z)^{n+1}},$$

unda

$$\operatorname{res}_{z=a_k} K_n(\xi, z) = \begin{cases} 0, n \geq 1, \\ 1, n = 1 \end{cases}$$

tenglik o'rinli bo'ladi.

Aytaylik $f(z)$ funksiya $z = a$ nuqtada Loran qatoriga yoyilgan bo'lsin:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k \psi^k(z, a) = \sum_{k=-\infty}^{\infty} c_k \left(z - a + \int_{\gamma(a, z)} \overline{A(\tau)} d\tau \right)^k.$$

5-teorema [9]. $A(z)$ – analitik $f(z)$ funksiya $a \in D$ yakkalangan maxsus nuqtasidagi chegirmasi Loran qatorining c_{-1} koefitsiyentining qiymatiga teng:

$$\operatorname{res}_{z=a} f(z) = c_{-1}. \quad (5)$$

2-ta'rif [9]. $z = a$ nuqta $A(z)$ – analitik funksiyaning n tartibli qutb nuqtasi bo'lsin, u holda

$z = a$ nuqta $\frac{1}{f(z)}$ funksiyaning tartibli noli deyiladi.

6-teorema [9]. $a \in \square$ yakkalangan maxsus nuqta $A(z)$ – analitik funksiyaning qutb maxsus nuqtasi bo‘lishi uchun funksiyaning a nuqtadagi $A(z)$ – lemniskata atrofida Loran qatoriga yoyilmasinig manfiy darajali hadlaridan chekli sonlagisi bo‘lishi zarur va yetarli:

$$f(z) = \sum_{k=-n}^{\infty} c_k \psi^k(z, a), \quad n \geq 1.$$

7-teorema [9]. Faraz qilaylik, $z = a$ nuqta $f(z)$ $A(z)$ – analitik funksiyaning n karrali qutb nuqtasi bo‘lsin. U holda quyidagi formula o‘rinli bo‘ladi:

$$\operatorname{res}_{z=a} f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{\partial^{n-1}}{\partial z^{n-1}} [f(z) \psi^n(z, a)] \quad (6)$$

2. $A(z)$ -analitik funksiyalar uchun ekstremum prinsipi va Koshi tipidagi integral tushunchasi

Aytaylik, $D \subset \square$ qavariq soha va bu sohada kompakt yotuvchi $L(a, r) \subset\subset D$ lemniskata berilgan bo‘lsin. U holda $f(z) \in O_A(L(a, r)) \cap C(\bar{L}(a, r))$ funksiya uchun, shuningdek (3) formulaga ko‘ra quyidagi munosabat o‘rinli bo‘ladi:

$$f(z) = \int_{|\psi(\zeta, z)|=r} f(\zeta) K(\zeta, z) (d\zeta + A(\zeta) d\bar{\zeta}). \quad (7)$$

Bu ifodada Eyler almashtirishini kiritib, (7) formula quyidagi ko‘rinishga keladi:

$$\begin{aligned} \psi(\zeta, z) &= re^{it}, d\psi(\zeta, z) = rie^{it} dt, 0 \leq t < 2\pi \Rightarrow \\ f(z) &= \frac{1}{2\pi i} \int_{|\psi(\zeta, z)|=r} \frac{f(\zeta)}{\psi(\zeta, z)} d\psi(\zeta, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{it})}{re^{it}} rie^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) dt. \end{aligned} \quad (8)$$

Oxirgi tenglikdan quyidagi xossani ta’kidlash mumkin:

1-tasdiq. $f(z) \in O_A(D)$ funksiyaning D sohaning ixtiyoriy z nuqtasidagi qiymati uning markazi z nuqtada bo‘lgan $L(z, r)$ lemniskataning chegarasi bo‘yicha olingan o‘rta qiymat integraliga teng bo‘ladi.

Endi quyidagi belgilash kiritamiz:

$$\max_{\zeta \in \partial L(z, r)} |f(\zeta)| = M(r).$$

U holda (8) formuladan quyidagi munosabat o‘rinli bo‘ladi:

$$|f(z)| \leq M(r). \quad (9)$$

$f(\zeta)$ funksiyaning $|\psi(z, a)| = r$ yopiq konturda uzluksizligidan bu konturning biror nuqtasidagi qiymati $M(r)$ ga erishadi. Chunki lemniskataning radiusi yetarlicha kichik qilib tanlansa, u holda (9) tengsizlikdan kelib chiqadiki, ixtiyoriy $z \in D$ nuqtaning atrofida boshqa nuqta aniqlanib, unda bu nuqtadagi funksiya modulidan kam bo‘lmaydi. Shunday qilib, D sohada $A(z)$ – analitik funksiyaning moduli sohaning bitta nuqtasida qat’iy maksimumga ega bo‘lmaydi. Bu tasdiqdan xulosalaymizki, ta’riflangan modulning maksimum prinsipi qo‘shimcha isboti mavjud va agar $f(z) \neq \text{const}$ bo‘lsa, u holda modulning noqat’iy maksimumi sohaning ichki nuqtasida erishmaydi.

Yuqoridagi tasdiqlardan quyidagi muhim teoremani keltiramiz:

8-teorema. Agar $f(z) \neq \text{const}$ funksiya $A(z)$ – analitik bo‘lsa, u holda funksiya modulining yuqori chegarasi D sohaning birorta ham nuqtasida erishmaydi.

Isbot. Haqiqatan ham $M = \sup_{z \in D} |f(z)|$ bo‘lsin. Agar $M = \infty$ bo‘lsa, u holda teoremaning natijasi ravshan bo‘ladi. Shuningdek, $M = 0$ bo‘lsa, u holda $f(z) \equiv 0$ bo‘ladi. Aytaylik, $0 < M < \infty$ bo‘lsin,

aniqlik uchun $z_0 \in D$ uchun $M = |f(z_0)|$ deb olaylik. U vaqtda (8) formuladan quyidagi munosabat o‘rinli bo‘ladi:

$$M = |f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(\rho, it_0) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\rho, it_0)| dt, \quad (10)$$

bu yerda ρ - ixtiyoriy musbat son bo‘lib, z_0 nuqtaning biror atrofi bo‘lgan lemniskataning chegarasi $l = \{z : |\psi(z, z_0)| = \rho\}$ yopiq egri chiziqdan iboratdir. Ammo $|f(\rho, it_0)| \leq M$ bo‘ladi va agar l ning biror nuqtasida $m = |f(\rho, it_0)| < M$ qat’iy tengsizlik o‘rinli bo‘lsa, u holda funksiya modulining uzluksizligidan t qiymatni o‘z ichiga oladigan uzunligi $\delta > 0$ bo‘lgan butun segment σ mavjud bo‘ladiki, bunda quyidagi tengsizlik bajariladi: $|f(\rho, it_0)| \leq \frac{M+m}{2}$. Ammo bundan kelib chiqadiki,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\rho, it_0)| dt = \frac{1}{2\pi} \int_{[0, 2\pi] \setminus \sigma} |f(\rho, it_0)| dt + \frac{1}{2\pi} \int_{\sigma} |f(\rho, it_0)| dt \leq \frac{M(2\pi - \delta)}{2\pi} + \frac{(M+m)\delta}{4\pi} < M,$$

bu tengsizlik (10) tengsizlikka zid keladi.

Shunday qilib, ixtiyoriy $0 \leq t < 2\pi$ va ixtiyoriy yetarlicha kichik ρ lar uchun $|f(\rho, it)| = M$ tenglik o‘rinli bo‘ladi.

Boshqacha aytganda, tasdig‘imizdan $A(z)$ – analitik funksiya $f(z)$ modulining yuqori chegarasiga biror $z_0 \in D$ nuqtada erishadi, bundan kelib chiqadiki, funksiya z_0 nuqta atrofining barcha nuqtalarida maksimumga erishadi (qaralayotgan z_0 nuqta atrofi D sohasida kompakt yotadi). ■

$f(z) \neq \text{const}$ funksiya yopiq $\bar{D}$ sohada uzluksiz va D sohada $A(z)$ – analitik bo‘lsin. U holda funksiya moduli $\bar{D}$ sohada uzluksiz bo‘ladi va bu sohaning biror $\zeta \in \bar{D}$ nuqtasida yuqori chegarasiga erishadi. Shuningdek, bu nuqta D sohaning chegaraviy nuqtasi bo‘ladi.

1-natija. Agar $f(z) \neq \text{const}$ $\bar{D}$ sohada uzluksiz va D sohada $A(z)$ – analitik funksiya bo‘lsa, u holda funksiya moduli sohaning chegaraviy nuqtasida o‘zining quyi chegarasiga erishadi.

Yuqorida xossalar $A(z)$ – analitik funksiyalarning ekstremum prinsipi deyiladi.

Funksiyalar nazariyasida Koshi integralining umumlashmasi muhim rol o‘ynaydi va Koshi tipidagi integral deyiladi.

3-ta’rif. Aytaylik, $l \subset \subset D$ silliq egri chiziq va $f(z) \in O_A(D) \cap C(\bar{D})$ funksiya berilgan bo‘lsin. U vaqtda

$$\int_l f(z) K(z, \zeta) (dz + A(z) d\bar{z}) \quad (11)$$

integral mavjud va chekli bo‘lib, $A(z)$ – analitik funksiyalar uchun Koshi tipidagi integral deyiladi, bu yerda $K(z, \zeta)$ Koshi yadrosi.

Koshi tipidagi integrallar yordamida juda ko‘plab masalalar analitik usulda yechilgan. Jumladan, aynan Koshining integral formulasi, Loran qatorlarining koeffitsiyentlari, chegirmalar Koshi tipidagi integrallar orqali ifodalanadi.

$A(z)$ -analitik funksiyalarning Koshi tipidagi integrallarga va chegirmalar hisoblashga doir tatbiqlari.

1-misol. $f(z)$ funksiya $L(a, r)$ lemniskatada $A(z)$ – analitik bo‘lsa va chegaralangan bo‘lsa, u holda ushbu $L(a, r)$ dan olingan ixtiyoriy ikkita nuqta z_1 va z_2 uchun quyidagi munosabat

$$\left| \int_{\gamma(z_1, z_2)} f(\xi) (d\xi + A(\xi) d\bar{\xi}) \right| \leq M |\psi(z_1, z_2)|$$

bajarilishini isbotlang, bu yerda M biror musbat son bo'lsin.

Isbot. Dastlab, $f(z) \in O_A(L(a, r))$ funksiya chegaralanganligini ko'rsatsak,

$$\exists M > 0, |f(z)| \leq M$$

shart bajariladi. Endi $\forall z_1, z_2 \in L(a, r)$ nuqtalar olib, $f(z)$ funksiyadan integralini baholab, yuqoridagi munosabat keltirib chiqaramiz:

$$\left| \int_{\gamma(z_1, z_2)} f(\xi) (d\xi + A(\xi) d\bar{\xi}) \right| \leq \int_{\gamma(z_1, z_2)} |f(\xi)| \cdot |d\xi + A(\xi) d\bar{\xi}| \leq M \int_{\gamma(z_1, z_2)} |d\xi + A(\xi) d\bar{\xi}| = M |\psi(z_1, z_2)| = M \cdot B,$$

bu yerda B son o'zgarmas qiymatni ifodalaydi.

2-misol. Quyidagi funksiyalarni boshlang'ich funksiyaga ega emasligini ko'rsating.

$$1) f(z) = \frac{1}{\psi(z, a)}, U_\varepsilon = \{z : |\psi(z, a)| < \varepsilon\}, \partial U_\varepsilon = \gamma_\varepsilon. \quad 2) f(z) = \frac{1}{\psi(z, a)} - \frac{1}{\psi(z, a) - r},$$

$$(0 < |\psi(z, a)| < r), \quad \gamma_\varepsilon^1 = \{z : |\psi(z, a)| < \varepsilon\}, \quad \gamma_\varepsilon^2 = \{z : |\psi(z, a) - r| < \varepsilon\}.$$

$$3) f(z) = \frac{1}{\psi(z, a)(1 - \psi^2(z, a))} \quad (0 < |\psi(z, a)| < r), \quad \gamma_\varepsilon^1 = \{z : |\psi(z, a)| < \varepsilon\},$$

$$\gamma_\varepsilon^2 = \{z : |\psi(z, a) - r| < \varepsilon\}, \quad \gamma_\varepsilon^3 = \{z : |\psi(z, a) + r| < \varepsilon\}.$$

Yechim. Berilgan funksiyalarni qaralayotgan soha chegarasi bo'yicha boshlang'ich funksiyasini mavjud bo'lmasligini ko'rsatish uchun uni chegaralanmagan bo'lishini ko'rsatishimiz yetarli:

$$1) \int_{\gamma_\varepsilon} \frac{dz + A(z)d\bar{z}}{\psi(z, a)} = \int_{\gamma_\varepsilon} \frac{d\psi(z, a)}{\psi(z, a)} = \ln |\psi(z, a)| \Big|_{\gamma_\varepsilon} < \ln \varepsilon \xrightarrow{\varepsilon \rightarrow 0} -\infty.$$

$$2) \int_{\gamma_\varepsilon^1} \left(\frac{1}{\psi(z, a)} - \frac{1}{\psi(z, a) - r} \right) (dz + A(z)d\bar{z}) = \int_{\gamma_\varepsilon^1} \left(\frac{d\psi(z, a)}{\psi(z, a)} - \frac{d\psi(z, a)}{\psi(z, a) - r} \right) \Big|_{\gamma_\varepsilon^1} =$$

$$= \ln \left| \frac{\psi(z, a)}{\psi(z, a) - r} \right| \Big|_{\gamma_\varepsilon^1} \leq \ln \left| \frac{\varepsilon}{\varepsilon - r} \right| \xrightarrow{\varepsilon \rightarrow 0} -\infty.$$

$$3) \int_{\gamma_\varepsilon^1} \left(\frac{1}{\psi(z, a)} + \frac{1}{2(1 - \psi(z, a))} - \frac{1}{2(1 + \psi(z, a))} \right) d\psi(z, a) =$$

$$= \ln \left(|\psi(z, a)| + \frac{1}{2} \ln \frac{1 - |\psi(z, a)|}{1 + |\psi(z, a)|} \right) \Big|_{\gamma_\varepsilon^1} \leq \ln \varepsilon + \frac{1}{2} \ln \frac{1 + \varepsilon}{1 - \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} -\infty.$$

3-misol. $I = \int_{|\psi(a, z)=2r|} \frac{dz + A(z)d\bar{z}}{\psi^2(a, z) + 1}$ integral hisoblang.

Yechim: $f(z) \in O_A(G) \cap C(\bar{G})$ berilgan bo'lsin.

Integralni hisoblash uchun quyidagi almashtirishlarni kiritamiz:

$$\psi(a, z) = re^{i\varphi}, \quad dz + A(z)d\bar{z} = d\psi(a, z) = rie^{i\varphi} d\varphi.$$

Integralga belgilangan ifodallarni qo'yamiz.

$$I = \int_0^{2\pi} \frac{re^{i\varphi} d\varphi}{r^2 e^{2i\varphi} + 1} = re^{i\varphi} \int_0^{2\pi} \frac{re^{i\varphi} d\varphi}{r^2 e^{2i\varphi} + 1}$$

$[e^{i\varphi} = t]$ deb belgilab, $I = \frac{1}{r^2} \ln(r^2 t^2 + 1)$ integralni tuzamiz va uni hisoblaymiz:

$$\frac{1}{r^2} \cdot ri \cdot \ln(r^2 t^2 + 1) \Big|_0^{2\pi} = \frac{i}{4r} (\ln(r^2 t^2 + 1) - \ln(r^2 t^2 + 1)) = 0.$$

4-misol. $f(z) = \frac{1}{e^{\psi(-\pi i, z)}}$ funksiyaning maxsus nuqtadagi chegirmasi hisoblang.

Yechim. $f(z) \in O_A(L(0, r))$ berilgan bo'lsin. $e^{\psi(-\pi i, z)}$ funksiyaning Loran qatoriga yoyamiz va qatorning C_{-1} hadni ajratib olamiz:

$$\begin{aligned} \frac{1}{\sum_{n=0}^{\infty} \frac{\psi^n(-\pi i, z)}{n!}} &= \frac{1}{\psi(-\pi i, z) + \frac{\psi^2(-\pi i, z)}{2!} + \dots + \frac{\psi^n(-\pi i, z)}{n!}} = \\ &= \frac{1}{\psi(-\pi i, z)} \frac{1}{1 + \frac{\psi^2(-\pi i, z)}{2!} + \dots + \frac{\psi^{n-1}(-\pi i, z)}{n!}}. \end{aligned}$$

(5) munosabatga ko'ra chegirmani hisoblaymiz:

$$f(z) = \operatorname{res}_{z=-\pi i} f(z) = \frac{1}{1+0} = 1.$$

5-misol. $f(z) = \frac{1}{\psi^3(0, z) \cdot \psi(1, z) \cdot \psi(-1, z)}$ funksiyaning maxsus nuqtalaridagi chegirmasini hisoblang.

hisoblang.

Xulosa. *Yechim.* $f(z) \in O_A(L(0, r))$ berilgan bo'lsin. $z=0$ nuqta $f(z) = \frac{1}{\psi(1, z) \cdot \psi(-1, z)}$

funksiyaning 3-tartibli qutb maxsus nuqtasi bo'lsin. U holda 7-teorema ko'ra uning 1- va 2- tartibli hosilalarini hisoblaymiz:

$$f''(z) = \left(\frac{1}{\psi(1, z) \cdot \psi(-1, z)} \right) \left(\frac{1}{\psi^2(1, z)} + \frac{1}{\psi^2(-1, z)} + \frac{1}{\psi(1, z) \cdot \psi(-1, z)} \right)$$

ifodani (6) ga qo'yib chegirmani topamiz:

$$\operatorname{res}_{z=0} f(z) = \frac{1}{2!} \cdot \left(\frac{1}{\psi(1, 0) \cdot \psi(-1, 0)} \right) \left(\frac{1}{\psi^2(1, 0)} + \frac{1}{\psi^2(-1, 0)} + \frac{1}{\psi(1, 0) \cdot \psi(-1, 0)} \right).$$

$z=1$ nuqta $f(z) = \frac{1}{\psi^3(0, z) \cdot \psi(-1, z)}$ funksiyaning 1-tartibli qutb maxsus nuqtasi bo'ladi. U

holda (5) ga ko'ra

$$\operatorname{res}_{z=1} f(z) = \frac{1}{\psi^3(0, 1) \cdot \psi(-1, 1)}.$$

$z=-1$ nuqta $f(z) = \frac{1}{\psi^3(0, z) \cdot \psi(1, z)}$ funksiyaning 1-tartibli qutb maxsus nuqtasi bo'ladi. U

holda (5) ga ko'ra

$$\operatorname{res}_{z=-1} f(z) = \frac{e^{\psi(0,-1)}}{\psi^3(0,-1) \cdot \psi(1,-1)}$$

bo'ladi.

ADABIYOTLAR:

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ON DYNAMICS OF A SEPARABLE QUADRATIC STOCHASTIC OPERATOR IN S^2

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Abstract. In this paper, we study a class of separable quadratic stochastic operators defined on a finite-dimensional simplex. That is, we consider cubic stochastic operators defined as a product of two linear operators defined on a simplex. It is shown that for a separable quadratic stochastic operator defined on a two-dimensional simplex, the vertices of the simplex are fixed points. It is also proven that the orbit of a separable quadratic stochastic operator converges for any starting point taken from the simplex.

Keywords: quadratic stochastic operator; separable quadratic stochastic operator; orbit, simplex.

О ДИНАМИКЕ РАЗДЕЛИМОГО КВАДРАТИЧНЫЙ СТОХАСТИЧЕСКОГО ОПЕРАТОРА В S^2

Аннотация. В данной работе изучается класс сепарабельных квадратичный стохастических операторов, заданных на конечномерном симплексе. То есть, рассматриваются квадратичный стохастические операторы, заданные как произведение двух линейных операторов, заданных на симплексе. Показано, что для сепарабельного квадратичный стохастического оператора, заданного на двумерном симплексе, вершины симплекса являются неподвижными точками.

Ключевые слова: Квадратичный стохастический оператор; сепарабел квадратичный стохастический оператор; орбита, симплекс.

S^2 DA ANIQLANGAN SEPARABEL KVADRATIK STOXASTIK OPERATORNING DINAMIKASI HAQIDA

Annotatsiya. Ushbu maqolada ikki o'lchamli simpleksda aniqlangan separabel kvadratik stoxastik operatorning dinamikasi o'rganilgan. Ya'ni, simpleksda aniqlangan ikkita chiziqli operatorlarning ko'paytmasi sifatida aniqlangan kvadratik stoxastik operatorlarni ko'rib chiqamiz. Ikki o'lchamli simpleksda aniqlangan separabel kvadratik stoxastik operator uchun simpleksning uchlari qo'zg'almas nuqtalar ekanligi ko'rsatilgan. Shuningdek, separabel kvadratik stoxastik operatorning orbitasi simpleksdan olingan har qanday boshlang'ich nuqta uchun yaqinlashuvchi ekanligi isbotlangan.

Kalit so'zlar: kvadratik stoxastik operator; separabel kvadratik stoxastik operator; orbita, simpleks.

Many systems can be represented using nonlinear operators. Among the simplest nonlinear examples is the quadratic stochastic operator (QSO). This operator models the evolution of a population and was originally introduced by Bernstein in [1].

For more than 80 years, the theory of QSOs has been developed and many papers were published (see e.g. [4]-[8], [16]-[17]). In recent years it has again become of interest in connection with its numerous applications in many branches of mathematics, biology and physics.

Let $E = \{1, 2, \dots, m\}$ be a finite set and the set of all probability distribution on E

$$S^{m-1} = \left\{ x = (x_1, \dots, x_m) \in \square^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\} \quad (1)$$

be the $(m-1)$ -dimensional simplex. A QSO is a mapping defined as $V : S^{m-1} \rightarrow S^{m-1}$ of the simplex into itself, of the form $V(x) = x' \in S^{m-1}$, where

$$x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k \in E, \quad (2)$$

and the coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} = P_{ji,k} \geq 0, \quad \sum_{k=1}^m P_{ij,k} = 1 \quad \text{for all } i, j \in E. \quad (3)$$

The trajectory (orbit) $\{\mathbf{x}^{(n)}\}_{n \geq 0}$, of V for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{(n+1)}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. This problem was solved completely for the Volterra QSO.

The operator V is called Volterra QSO, if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$, $i, j, k \in E$. For the Volterra QSO the general formula was given in [4],

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad (4)$$

where $a_{ki} = 2P_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. Moreover, $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$ for all $i, k \in E$.

In [4], the theory of Volterra QSO was developed using theory of the Lyapunov functions and tournaments. But non-Volterra QSOs were not completely studied. Because, there is no general theory that can be applied for study of non-Volterra operators. In [16] Separable Quadratic Stochastic Operators (SQSOs) were introduced. Volterra QSO (4) has a form as SQSO, but in [17] it was proved that it coincides with a SQSO if and only if it is a linear operator.

In this paper we consider another class of quadratic operators which we call separable quadratic stochastic operators [11-13].

In Section 2, we recall the definition of QSOs and definitions and known results. In Section 3 for a SQSO defined on the two-dimensional simplex, we prove that it has two fixed points and we find the boundary ∂S^2 is an invariant set.

Preliminaries and known results.

Separable quadratic stochastic operator. Let us recall some necessary definition and notations. In [18] Separable Quadratic Stochastic Operators were introduced as follows: The QSO (2), (3) with additional condition

$$P_{ij,k} = a_{ik} b_{jk} \quad \text{for all } i, j, k \in E \quad (5)$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices $A = (a_{ik})$ and $B = (b_{jk})$ such that the conditions (3) are satisfied for the coefficients (5).

Then the QSO V corresponding to the coefficients (5) has the form.

$$x'_k = (V(\mathbf{x}))_k = (A(\mathbf{x}))_k (B(\mathbf{x}))_k, \quad (6)$$

where $(A(\mathbf{x}))_k = \sum_{i=1}^m a_{ik} x_i$, $(B(\mathbf{x}))_k = \sum_{j=1}^m b_{jk} x_j$.

Definition 1. [16] The QSO (6) is called separable quadratic stochastic operator (SQSO).

Quadratic stochastic operator. The QSO is a mapping $V: S^{m-1} \rightarrow S^{m-1}$ of the form

$$x'_i = \sum_{j=1}^m P_{ij,k} x_j, \quad k \in (1, 2, \dots, m) \quad (7)$$

where $P_{ij,k}$ are coefficients of heredity such that

$$P_{ij,k} \geq 0, \quad \sum_{k=1}^m P_{ij,k} = 1, \quad \forall i, j, k \in E. \quad (8)$$

and we suppose that the coefficients $P_{ij,k}$ do not change for any permutation of i, j .

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}_{n \geq 0}$ of initial point $\mathbf{x}^{(0)}$ under action of QSO (8) is defined by $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \dots$ with $\mathbf{x} = \mathbf{x}^{(0)}$. Denote by $\omega(\mathbf{x}^{(0)})$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$. Since $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is a compact set, it follows that $\omega(\mathbf{x}^{(0)}) \neq \emptyset$. If $\omega(\mathbf{x}^{(0)})$ consists of a single point, then the trajectory converges and $\omega(\mathbf{x}^{(0)})$ is a fixed point of the operator

V . A point $\mathbf{x} \in S^{m-1}$ is called a fixed of the V if $V(\mathbf{x}) = \mathbf{x}$. Denote by $\text{Fix}(V)$ the set of all fixed points of the operator V , i.e.

$$\text{Fix}(V) = \{\mathbf{x} \in S^{m-1} : V(\mathbf{x}) = \mathbf{x}\}.$$

Let $DV(\mathbf{x}^*) = (\partial V_i / \partial x_j)(\mathbf{x}^*)$ be a Jacobian of V at the point $\mathbf{x}^*$.

Definition 2 ([3]): A fixed point $\mathbf{x}^*$ is called hyperbolic if its Jacobian $DV(\mathbf{x}^*)$ has no eigenvalues on the unit circle in $\mathbb{C}$.

Definition 3 ([3]): A hyperbolic fixed point $\mathbf{x}^*$ is called:

- (i) attracting if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are in the unit disk;
- (ii) repelling if all the eigenvalues of the Jacobian $DV(\mathbf{x}^*)$ are outside the closed unit disk;
- (iii) a saddle otherwise;

Main result. In this section we consider QSO (7), (8) with additional condition

$$P_{ij,k} = a_{ik} b_{jk}, \quad \text{for all } i, j, k \in E, \quad (9)$$

where $a_{ik}, b_{jk} \in \mathbb{R}$ entries of matrices $A = (a_{ik})$, $B = (b_{jk})$ such that the conditions (8) are satisfied for the coefficients (9).

Then the QSO V corresponding to the coefficients (9) has the form

$$x'_k = (V(\mathbf{x}))_k = (A(\mathbf{x}))_k (B(\mathbf{x}))_k \quad (10)$$

where $(A(\mathbf{x}))_k = \sum_{i=1}^m a_{ik} x_i$, $(B(\mathbf{x}))_k = \sum_{j=1}^m b_{jk} x_j$.

Definition 4. [11] The QSO (10) is called separable quadratic stochastic operator (SQSO).

Let us consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1+a & 1 \\ 1-a & 1 & 1+b \\ 1 & 1-b & 1 \end{pmatrix}, \quad (11)$$

where $a, b \in [-1; 1]$.

Then corresponding SQSO $V : S^2 \rightarrow S^2$ is:

$$V : \begin{cases} x'_1 = x_1^2 + (1-a)x_1x_2 + x_1x_3 \\ x'_2 = x_2^2 + (1+a)x_1x_2 + (1-b)x_2x_3 \\ x'_3 = x_3^2 + x_1x_3 + (1+b)x_2x_3 \end{cases} \quad (12)$$

Using the equation $x_1 + x_2 + x_3 = 1$ we rewrite the operator (21) as follows

$$V : \begin{cases} x'_1 = x_1(1-ax_2) \\ x'_2 = x_2(1+ax_1-bx_3) \\ x'_3 = x_3(1+bx_2) \end{cases} \quad (13)$$

If $a = 0$ and $b = 0$ the SQSO (22) is the identity map. For this below, we consider the case when $a \neq 0$ and $b \neq 0$. Let a face of the simplex S^2 be the set $\Gamma_\alpha = \{\mathbf{x} \in S^2 : x_i = 0, i \notin \alpha \subset \{1, 2, 3\}\}$. Let the set $\text{int } S^2 = \{\mathbf{x} \in S^2 : x_1x_2x_3 > 0\}$ and let the set $\partial S^2 = S^2 \setminus \text{int } S^2$ be the interior and the boundary of the simplex S^2 respectively. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ be the vertexes of the two-dimensional simplex.

Theorem 2. For the SQSO V (22), the following assertions true:

- (i) The faces $\Gamma_{\{1,2\}}$, $\Gamma_{\{2,3\}}$, $\Gamma_{\{1,3\}}$ of the simplex S^2 are invariant sets;
- (ii) $\text{Fix}(V) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
- (iii) The fixed point $\mathbf{e}_1$ is a hyperbolic point.

$$e_2 \text{ is a/an } \begin{cases} \text{repelling,} & \text{if } a \in [-1, 0) \quad b \in (0, 1] \\ \text{saddle,} & \text{if } a \in [-1, 0) \quad b \in [-1, 0) \\ \text{saddle,} & \text{if } a \in (0, 1] \quad b \in (0, 1] \\ \text{attracting,} & \text{if } a \in (0, 1] \quad b \in [-1, 0) \end{cases}$$

The fixed point e_3 is hyperbolic point.

Proof: (i) Obviously (ii) To find the fixed points we consider the equation $V(x) = x$, that is the following system of equations

$$\begin{cases} x_1 = x_1(1 - ax_2) \\ x_2 = x_2(1 + ax_1 - bx_3) \\ x_3 = x_3(1 + bx_2) \end{cases} \quad (14)$$

Let $x_1 = 0$. Then from the second equation of (23) it follows

$$x_2 = x_2(1 - bx_3) \Rightarrow x_2 = 0 \text{ and } 1 = 1 - bx_3$$

It is clear that if $x_2 = 0$ then we have $x_3 = 0$. If $x_2 > 0$ then from the last equation one has

$$x_3 = 1 \Rightarrow x_2 = 0$$

We take solution $x_3 = 1$, then it follows that $x_2 = 0$. Consequently, if $x_1 = 0$, we obtain the fixed points e_2, e_3 .

Similarly, in the case $x_2 = 0$ and $x_3 = 0$, we have the fixed points e_1, e_3 and e_1, e_2 , respectively.

(a) Suppose that $x_1 x_2 x_3 \neq 0$. Then from the system (23), one has

$$\begin{cases} 1 = 1 - ax_2 \\ 1 = 1 + ax_1 - bx_3 \\ 1 = 1 + bx_2 \end{cases} \quad (15)$$

We see that from the first equation $1 = 1 - ax_2 \Rightarrow x_2 = 0$, but we assumed that $x_2 \neq 0$.

Consequently, we have that $Fix(V) \cap \text{int } S^2 = \emptyset$

(iii) To find the type of a fixed point of the SQSO (25), we rewrite it in the form

$$\begin{cases} x'_1 = x_1(1 - ax_2) \\ x'_2 = x_2(1 + ax_1 - b(1 - x_1 - x_2)) \end{cases} \quad (16)$$

where $(x_1, x_2) \in \{(x, y) : x, y \geq 0, 0 \leq x + y \leq 1\}$ and x_1, x_2 are the first two coordinates of a point lying in the simplex S^2 . The Jacobian of the operator (26) at a fixed point e_1 has the following eigenvalues $\lambda_1 = 1, \lambda_2 = a + 1$, at the e_2 has the following eigenvalues $\lambda_1 = 1 - a, \lambda_2 = 1 + b$, and at the e_3 has the following eigenvalues $\lambda_1 = 1, \lambda_2 = 1 - b$. Therefore, it follows that:

If $a \in [-1, 0)$ and $b \in (0, 1]$, then

$$\begin{cases} e_1 & \text{is a nohyperbolic point} \\ e_2 & \text{is a repelling point} \\ e_3 & \text{is a nohyperbolic point} \end{cases}$$

If $a \in [-1, 0)$ and $b \in [-1, 0)$

$$\begin{cases} e_1 & \text{is a nohyperbolic point} \\ e_2 & \text{is a saddle point} \\ e_3 & \text{is a nohyperbolic point} \end{cases}$$

If $a \in (0, 1]$ and $b \in [-1, 0)$

$$\begin{cases} \mathbf{e}_1 \text{ is a nohyperbolic point} \\ \mathbf{e}_2 \text{ is a attracting point} \\ \mathbf{e}_3 \text{ is a nohyperbolic point} \end{cases}$$

If $a \in (0,1]$ and $b \in (0,1]$

$$\begin{cases} \mathbf{e}_1 \text{ is a nohyperbolic point} \\ \mathbf{e}_2 \text{ is a saddle point} \\ \mathbf{e}_3 \text{ is a nohyperbolic point} \end{cases}$$

The set of limit points. Let $\mathbf{x}^{(0)} \in S^2$ be an initial point. Then the trajectory of $\mathbf{x}^{(0)}$ is denoted by $\{\mathbf{x}^{(n)}\}_{n \geq 0}$ and is defined as $V(\mathbf{x}^{(n)}) = \mathbf{x}^{(n+1)}$ where $n = 0, 1, 2, \dots$.

Theorem 3. For the SQSO V (23), the following assertions true:

(i) If $a \in (0,1]$ and $b \in (0,1]$, then

$$\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_2, & \text{if } \mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \mathbf{e}_1 \\ \mathbf{e}_3, & \text{if } \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}, \end{cases},$$

(ii) If $a \in (0,1]$ and $b \in [-1,0)$, then

$$\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_2, & \text{if } \mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \mathbf{e}_3 \\ \mathbf{e}_2, & \text{if } \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}, \end{cases},$$

(iii) If $a \in [-1,0)$ and $b \in (0,1]$

$$\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_3, & \text{if } \mathbf{x}^{(0)} \in \Gamma_{\{2,3\}} \setminus \mathbf{e}_2 \\ \mathbf{e}_1, & \text{if } \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}, \end{cases},$$

(iv) If $a \in [-1,0)$ and $b \in [-1,0)$

$$\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \begin{cases} \mathbf{e}_2, & \text{if } \mathbf{x}^{(0)} \in \Gamma_{\{2,3\}} \setminus \mathbf{e}_3 \\ \mathbf{e}_1, & \text{if } \mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}, \end{cases},$$

Proof: (i) If $a \in (0,1]$ and $b \in (0,1]$. Let $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, 0) \in \Gamma_{\{1,2\}}$. By Theorem 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. For the SQSO V (22) is true $x_3^{(0)} = 0$, $n = 0, 1, 2, \dots$ at the face $\Gamma_{\{1,2\}}$. The restriction of the SQSO V (22) on the face

$$\Gamma_{\{1,2\}} \text{ has the form } \begin{cases} x'_1 = x_1(1 - ax_2) \\ x'_2 = x_2(1 + ax_1) \end{cases} \quad (17)$$

From the first equation of (26), one has

$$x_1^{(n+1)} = x_1^{(n)}(1 - ax_2^{(n)}) \leq x_1^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. Moreover, from the second equation of (26), one has

$$x_2^{(n+1)} = x_2^{(n)}(1 + ax_1^{(n)}) \leq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \mathbf{x}^* = (x_1^*, x_2^*, 0)$. Since $\mathbf{x}^*$ should be fixed point we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n \rightarrow \infty} V^n(\mathbf{x}^{(0)}) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \mathbf{e}_1$.

Let $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}$. Then from the third equation of operator (22), one has

$$x_3^{(n+1)} = x_3^{(n)} (1 + bx_2^{(n+1)}) \geq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

So we that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. Moreover, from the second equation of operator (22) one

has
$$x_2^{(n+1)} = x_2^{(n)} (1 + ax_1^{(n)}) \leq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. This, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. So there is the limit $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = \mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$. Since $\mathbf{x}^*$ should be fixed point we get $\mathbf{x}^* = \mathbf{e}_3$. Consequently, we have that $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{e}_3$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}$.

(ii) If $a \in (0, 1]$ and $b \in [-1, 0)$. Let $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, 0) \in \Gamma_{\{1,2\}}$. By Theorem 1, the face $\Gamma_{\{1,2\}}$ is an invariant set and the vertexes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed points belonging to this face. For the SQSO V (22) is true $x_3^{(0)} = 0, n = 0, 1, 2, \dots$ at the face $\Gamma_{\{1,2\}}$. The restriction of the SQSO V (22) on the face $\Gamma_{\{1,2\}}$ has the form

$$\begin{cases} x_1' = x_1 (1 - ax_2) \\ x_2' = x_2 (1 + ax_1) \end{cases} \quad (18)$$

From the first equation of (27), one has

$$x_1^{(n+1)} = x_1^{(n)} (1 - ax_2^{(n)}) \leq x_1^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. Moreover, from the second equation of (22),

one has
$$x_2^{(n+1)} = x_2^{(n)} (1 + ax_1^{(n)}) \geq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. So there is the limit

$\lim_{n \rightarrow \infty} V^n(x^{(0)}) = \mathbf{x}^* = (x_1^*, x_2^*, 0)$. Since $\mathbf{x}^*$ should be fixed point we get $\mathbf{x}^* = \mathbf{e}_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in \Gamma_{\{1,2\}} \setminus \mathbf{e}_1$.

Let $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}$. Then from the third equation of operator (22), one has

$$x_3^{(n+1)} = x_3^{(n)} (1 + bx_2^{(n+1)}) \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

So we that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. Moreover, from the second equation of operator (22) one

has
$$x_2^{(n+1)} = x_2^{(n)} (1 + ax_1^{(n)}) \geq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. This, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. So there is the limit $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = \mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$. Since $\mathbf{x}^*$ should be fixed point we get $\mathbf{x}^* = \mathbf{e}_2$. Consequently, we have that $\lim_{n \rightarrow \infty} V^n(\mathbf{x}) = \mathbf{e}_2$ for any $\mathbf{x}^{(0)} \in S^2 \setminus \Gamma_{\{1,2\}}$.

(iii) If $a \in [-1, 0)$ and $b \in (0, 1]$ Let $\mathbf{x}^{(0)} = (0, x_2^{(0)}, x_3^{(0)}) \in \Gamma_{\{2,3\}}$. By Theorem 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes $\mathbf{e}_2$ and $\mathbf{e}_3$ are fixed points belonging to this face. For the SQSO V (22) is true $x_1^{(0)} = 0, n = 0, 1, 2, \dots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SQSO V (22) on the face $\Gamma_{\{2,3\}}$ has the

$$\text{form} \quad \begin{cases} x'_2 = x_2(1 - bx_3) \\ x'_3 = x_3(1 + bx_2) \end{cases}$$

(19) From the first equation of (28), one has

$$x_2^{(n+1)} = x_2^{(n)}(1 - bx_3^{(n)}) \leq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. Moreover, from the second equation of (28),

$$\text{one has} \quad x_3^{(n+1)} = x_3^{(n)}(1 + bx_2^{(n)}) \geq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. So there is the limit

$\lim_{n \rightarrow \infty} V^n(x^{(0)}) = x^* = (0, x_2^*, x_3^*)$. Since x^* should be fixed point we get $x^* = e_3$. That is, if $x_3^{(0)} > 0$ then we have that $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = e_3$ for any $x^{(0)} \in \Gamma_{\{2,3\}} \setminus e_2$.

Let $x^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}$. Then from the third equation of operator (22), one has

$$x_3^{(n+1)} = x_3^{(n)}(1 + bx_2^{(n+1)}) \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

So we that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. Moreover, from the second equation of operator (22) one

$$\text{has} \quad x_2^{(n+1)} = x_2^{(n)}(1 + ax_1^{(n)} - bx_3^{(n)}) \geq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. This, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. So there is the limit $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = x^* = (x_1^*, x_2^*, x_3^*)$. Since x^* should be fixed point we get $x^* = e_2$. Consequently, we have that $\lim_{n \rightarrow \infty} V^n(x) = e_2$ for any $x^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}$.

(iv) If $a \in [-1, 0)$ and $b \in [-1, 0)$ If $a \in [-1, 0)$ and $b \in (0, 1]$ Let $x^{(0)} = (0, x_2^{(0)}, x_3^{(0)}) \in \Gamma_{\{2,3\}}$.

By Theorem 1, the face $\Gamma_{\{2,3\}}$ is an invariant set and the vertexes e_2 and e_3 are fixed points belonging to this face. For the SQSO V (22) is true $x_1^{(0)} = 0$, $n = 0, 1, 2, \dots$ at the face $\Gamma_{\{2,3\}}$. The restriction of the SQSO

$$V(22) \text{ on the face } \Gamma_{\{2,3\}} \text{ has the form} \quad \begin{cases} x'_2 = x_2(1 - bx_3) \\ x'_3 = x_3(1 + bx_2) \end{cases} \quad (20)$$

From the first equation of (28), one has

$$x_2^{(n+1)} = x_2^{(n)}(1 - bx_3^{(n)}) \geq x_2^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. Moreover, from the second equation of (28),

$$\text{one has} \quad x_3^{(n+1)} = x_3^{(n)}(1 + bx_2^{(n)}) \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. So there is the limit

$\lim_{n \rightarrow \infty} V^n(x^{(0)}) = x^* = (0, x_2^*, x_3^*)$. Since x^* should be fixed point we get $x^* = e_2$. That is, if $x_2^{(0)} > 0$ then we have that $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = e_2$ for any $x^{(0)} \in \Gamma_{\{2,3\}} \setminus e_3$.

Let $x^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}$. Then from the third equation of operator (22), one has

$$x_3^{(n+1)} = x_3^{(n)}(1 + bx_2^{(n+1)}) \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

So we that there exists the $\lim_{n \rightarrow \infty} x_3^{(n)} = x_3^*$. Moreover, from the first equation of operator (22) one has

$$x_1^{(n+1)} = x_1^{(n)} (1 - \alpha x_2^{(n)}) \geq x_1^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore, it follows that there exists the $\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^*$. This, using $x_1^{(n)} + x_2^{(n)} + x_3^{(n)} = 1$ we have $\lim_{n \rightarrow \infty} x_2^{(n)} = x_2^*$. So there is the limit $\lim_{n \rightarrow \infty} V^n(x^{(0)}) = x^* = (x_1^*, x_2^*, x_3^*)$. Since x^* should be fixed point we get $x^* = e_1$. Consequently, we have that $\lim_{n \rightarrow \infty} V^{(n)}(x) = e_1$ for any $x^{(0)} \in S^2 \setminus \Gamma_{\{2,3\}}$.

The theorem is proved.

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IKKITA SILJISHLI NUQTADA KICHIK MAXSUSLIKKA EGA VOLTERRA TIPIDAGI INTEGRAL TENGLAMASI

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Annotatsiya. Ushbu maqolada ikkita siljishli nuqtaga ega hamda yadrosi kichik singulyarlikka ega bo‘lgan Volterra tipidagi integral tenglamalar o‘rganiladi. Avvalo, masalaning to‘g‘ri qo‘yilishi muhokama qilinib, integral tenglamaning funksional fazoda qanday ifodalanishi va uning yechimining mavjudligi hamda yagonaligi masalalari batafsil tahlil qilinadi. Maqolada integral operatorning xossalari va uning baholanishi, shuningdek, ketma-ket yaqinlashish usuli orqali yechimning qurilishi yoritiladi. Shu bilan birga, integral tenglamalarning yechimi va operatorning baholanishi o‘rtasidagi bog‘liqlik, shuningdek, singulyar yadrolar bilan ishlashning matematik xususiyatlari ham ko‘rib chiqiladi. Tadqiqot davomida yechimning konstruktiv qurilishi va uning funksional fazodagi barqarorligi, shuningdek, ketma-ket yaqinlashish usulining samaradorligi muhokama qilinadi.

Kalit so‘zlar: Volterra tenglamasi, siljishli yadro, kichik singulyarlik, nolokal shart, ketma-ket yaqinlashish usuli.

ИНТЕГРАЛЬНОЕ УРАВНЕНИЕ ТИПА ВОЛЬТЕРРЫ С МАЛОЙ ОСОБЕННОСТЬЮ В ДВУХ СМЕЩЁННЫХ ТОЧКАХ

Аннотация. В данной статье изучаются интегральные уравнения типа Вольтерры с двумя сдвигающимися точками и ядрами с малой сингулярностью. Сначала обсуждается правильная постановка задачи, её представление в функциональном пространстве, а также подробно анализируются вопросы существования и единственности решения. Рассматриваются свойства интегрального оператора и его оценка, а также построение решения методом последовательных приближений. Кроме того, анализируются взаимосвязь между решением и оценкой оператора, а также математические свойства работы с сингулярными ядрами. Также обсуждаются конструктивное построение решения, его устойчивость в функциональном пространстве и эффективность метода последовательных приближений.

Ключевые слова: уравнение Вольтерры, сдвигающееся ядро, малая сингулярность, нелокальное условие, метод последовательных приближений.

VOLTERRA-TYPE INTEGRAL EQUATION WITH SMALL SINGULARITY AT TWO SHIFT POINTS

Abstract. This article studies Volterra-type integral equations with two shift points and kernels with small singularity. First, the correct formulation of the problem is discussed, as well as its representation in a functional space, and the existence and uniqueness of the solution are analyzed in detail. The properties of the integral operator and its estimation, as well as the construction of the solution using the successive approximations method, are examined. Furthermore, the relationship between the solution and the operator estimation, as well as the mathematical properties of working with singular kernels, are considered. The constructive construction of the solution, its stability in the functional space, and the efficiency of the successive approximations method are also discussed.

Keywords: Volterra equation, shift kernel, small singularity, nonlocal condition, successive approximations method.

Kirish. Integral tenglamalar zamonaviy matematikaning turli sohalarida – fizika, mexanika, iqtisodiyot, biologiya va boshqa ilmiy tadqiqotlarda keng qo‘llaniladi. Ayniqsa, Volterra tipidagi integral tenglamalar vaqt bilan bog‘liq jarayonlarni, evolyutsion tizimlarni va harakat qonuniyatlarini matematik

model sifatida tavsiflashda muhim ahamiyatga ega. Bunday tenglamalarning o'rganilishi nafaqat nazariy jihatdan qiziqarli, balki amaliy hisob-kitoblar va ilmiy modellashda ham katta ahamiyatga ega.

Ushbu maqolada ikkita siljishli nuqtaga ega va yadrosi kichik singulyarlikka ega bo'lgan Volterra integral tenglamalari tahlil qilinadi. Bunday tenglamalar singulyar yadrolar bilan bog'liq maxsus xususiyatlarga ega bo'lib, yechimning mavjudligi, yagonaligi va barqarorligini o'rganishni talab qiladi. Shuningdek, ketma-ket yaqinlashish usuli orqali yechimlarni konstruktiv tarzda qurish imkoniyatlari ham ko'rib chiqiladi.

Adabiyotlar tahlili va metodologiya. Integral tenglamalar nazariyasi va ularning turli tipdagi masalalardagi qo'llanilishi matematikaning muhim bo'limi hisoblanadi. I.G.Petrovskiy o'zining "Лекции по теории интегральных уравнения" asarida integral tenglamalar nazariyasining asosiy tamoyillari va ularning funksional fazoda ifodalanishi haqida batafsil tushuntiradi. U yerda Volterra va Fredholm tipidagi integral tenglamalarning yechimlari, mavjudligi, yagonaligi va ularning konstruktiv qurilishi masalalari keng yoritilgan. Petrovskiyning tadqiqotlari integral tenglamalarni nazariy jihatdan mustahkamlash bilan birga, ularni amaliy hisob-kitoblar va turli ilmiy modellarda qo'llash imkoniyatini yaratadi [1].

So'nggi yillarda o'zbek olimlari ham integral va giperbolik tenglamalar bilan bog'liq masalalarni faol o'rganmoqdalar. M.Mirsaburov va S.Choriyeva tomonidan yozilgan "Об одной нелокальной краевой задаче для гиперболического уравнения, вырождающегося внутри области" maqolada ichki mintaqada singularlikka ega bo'lgan giperbolik tenglamalarning nolokal chegaraviy masalasini tahlil qilingan. Ularning natijalari shuni ko'rsatadiki, bunday masalalarda yechimni qurish va uning barqarorligini ta'minlash uchun maxsus metodlar, xususan, integral operatorning baholanishi va ketma-ket yaqinlashish usuli qo'llanilishi zarur. Ushbu yondashuv masalalarning yechimining mavjudligi va yagonaligini nazariy jihatdan kafolatlaydi va amaliy hisob-kitoblarda samarali natijalar beradi [2].

Muhokama va natijalar. Ushbu

$$v(x) = \int_x^1 \frac{K_1(x, s)v(s)ds}{|s - a_1 - b_1x|^l} + \int_x^1 \frac{K_2(x, s)v(s)ds}{|s - a_2 - b_2x|^l} + f(x), \quad \forall x \in I, \quad (1)$$

ko'rinishdagi tenglamani qaraymiz, bunda $l = 1 - \alpha + \beta < 1$.

Bu tenglama ikkita siljishli nuqtada kichik maxsuslikka ega "Volterra" tipidagi integral tenglama deyiladi.

Bu yerda

$$K_i(x, s) = \begin{cases} \frac{\mu_i \beta (1 - \alpha) (1 - x)^\beta b_i (\rho(s) - c)}{(a_i (1 - s))^\alpha (\rho(x) - c) (1 - a_i^{-\beta} \mu_1 - a_i^{-\beta} \mu_2)} \times \\ \times F(\alpha, 1 - \beta, 2; b_i (s - x) / a_i (1 - s)), & x \leq s \leq a_i + b_i x, \\ -\frac{\mu_i \Gamma(1 - \beta) (1 - \alpha) (a_i (1 - s))^{1 - \beta - \alpha} (1 - x)^\beta (\rho(s) - c)}{(b_i (s - x))^{1 - \beta} \Gamma(2 - \alpha - \beta) \Gamma(\alpha) (\rho(x) - c) (1 - a_i^{-\beta} \mu_1 - a_i^{-\beta} \mu_2)} \times \\ \times F(-\beta, 1 - \beta, 2 - \alpha - \beta; a_i (1 - s) / b_i (s - x)), & \\ a_i + b_i x \leq s \leq 1, & \end{cases}$$

bunda $a_i = \frac{2}{k_i + 1}$, $b_i = \frac{k_i - 1}{k_i + 1} = 1 - a_i$, $i = 1, 2$.

"Volterra" tipidagi (1) integral tenglama ketma-ket yaqinlashish usuli yordamida yechiladi.

Teorema. Agar yetarli kichik μ_1 va μ_2 larda

$$\left| \frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right| < 1$$

shart bajarilsa, tenglama yechimga ega, bunda $|(1 - x)^{\alpha - \beta} K_k(x, t)| \leq K_k$, $|f(x)| \leq M$, $k = 1, 2$

Shu bilan birgalikda

$$K_k(x, a_k + b_k x - 0) = \frac{\mu_k (1-x)^{\beta-\alpha} \Gamma(1-\alpha+\beta)}{a_k^\alpha b_k^\alpha \Gamma(1-\alpha) \Gamma(\beta)}, \quad k=1,2, \quad (2)$$

$$K_k(x, a_k + b_k x + 0) = -\frac{\mu_k (1-x)^{\beta-\alpha} \Gamma(1-\beta) \Gamma(1-\alpha+\beta)}{a_k^\alpha b_k^\alpha \Gamma^2(1-\alpha) \Gamma(\alpha)}, \quad k=1,2. \quad (3)$$

Quyidagi lemma o‘rinlidir.

Lemma. Funksiya $K_k(x, t) \in C^2(\bar{I}) \setminus \gamma_k$, $k=1,2$, bu funksiya $\gamma_k: t = a_k + b_k x$, egri chiziqda birinchi tur uzilishga ega, (1,1) nuqtadan tashqari, bu nuqtada ular $\alpha - \beta$ tartibdagi maxsuslikga ega. Bu lemmaning isboti (2) dan darhol kelib chiqadi [3].

(1) integral tenglamani o‘rganish.

(1) tenglama shu bilan qiziqarliki, u $a_1 + b_1 x$ va $a_2 + b_2 x$ chiziqda kichik yadroga ega

$x < a_1 + b_1 x < a_2 + b_2 x$, bunda $0 < a_k, b_k < 1$, $a_k + b_k = 1$, $k=1,2$.

(1) tenglamani yechish uchun ketma-ket yaqinlashish usuli bilan foydalanamiz [1]. Funktsional ketma-ketlikning $v_0(x)$, $v_1(x)$, ..., $v_n(x)$, ... hadlarni topish uchun quyidagi rekurrent munosabatni tuzamiz

$$v_n(x) = \int_x^1 \left(\frac{\mu_1 K_1(x, t)}{|t - a_1 - b_1 x|^\ell} + \frac{\mu_2 K_2(x, t)}{|t - a_2 - b_2 x|^\ell} \right) v_{n-1}(t) dt + f(x). \quad (4)$$

Nolinchi yaqinlashish sifatida $v_0(x) = f(x)$ ni olamiz. Bu holda birinchi yaqinlashish, quyidagi ko‘rinishga ega

$$v_1(x) = \int_x^1 \left(\frac{\mu_1 K_1(x, t)}{|t - a_1 - b_1 x|^\ell} + \frac{\mu_2 K_2(x, t)}{|t - a_2 - b_2 x|^\ell} \right) v_0(t) dt + f(x). \quad (5)$$

Bundan osonlikcha quyidagi baholashni topamiz:

$$\begin{aligned} |v_1(x) - v_0(x)| &\leq |\mu_1| \int_x^1 \frac{|K_1(x, t)| |f(x)| dt}{|t - a_1 - b_1 x|^\ell} + |\mu_2| \int_x^1 \frac{|K_2(x, t)| |f(x)| dt}{|t - a_2 - b_2 x|^\ell} \leq \\ &\leq |\mu_1| K_1 M (1-x)^{\beta-\alpha} \int_x^1 \frac{dt}{|t - a_1 - b_1 x|^\ell} + |\mu_2| K_2 M (1-x)^{\beta-\alpha} \int_x^1 \frac{dt}{|t - a_2 - b_2 x|^\ell} = \\ &= |\mu_1| K_1 M (1-x)^{\beta-\alpha} \left(\int_x^{a_1+b_1x} \frac{dt}{(a_1+b_1x-t)^\ell} + \int_{a_1+b_1x}^1 \frac{dt}{(t-a_1-b_1x)^\ell} \right) + \\ &+ |\mu_2| K_2 M (1-x)^{\beta-\alpha} \left(\int_x^{a_2+b_2x} \frac{dt}{(a_2+b_2x-t)^\ell} + \int_{a_2+b_2x}^1 \frac{dt}{(t-a_2-b_2x)^\ell} \right) = \\ &= |\mu_1| K_1 M (1-x)^{\beta-\alpha} 2^{1-\ell} \frac{a_1^{1-\ell} + b_1^{1-\ell}}{1-\ell} + |\mu_2| K_2 M (1-x)^{\beta-\alpha} 2^{1-\ell} \frac{a_2^{1-\ell} + b_2^{1-\ell}}{1-\ell}, \end{aligned}$$

bunda $|(1-x)^{\alpha-\beta} K_k(x, t)| \leq K_k$, $|f(x)| \leq M$.

Shunday qilib,

$$|v_1(x) - v_0(x)| \leq M \left(\frac{|\mu_1| K_1 (a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2 (a_2^{1-\ell} + b_2^{1-\ell})}{1-\ell} \right). \quad (6)$$

Endi ikkinchi yaqinlashishni qo‘llaymiz, (4) dan $n=2$ bo‘lganida

$$\nu_2(x) = \int_x^1 \left(\frac{\mu_1 K_1(x, t)}{|t - a_1 - b_1 x|^\ell} + \frac{\mu_2 K_2(x, t)}{|t - a_2 - b_2 x|^\ell} \right) \nu_1(t) dt + f(x) \quad (7)$$

(4) ni (agar $n = 2$ bo'lsa) hisoblab, (5) ni ayiramiz:

$$|\nu_2(x) - \nu_1(x)| \leq \int_x^1 \left(\frac{|\mu_1| |K_1(x, t)|}{|t - a_1 - b_1 x|^\ell} + \frac{|\mu_2| |K_2(x, t)|}{|t - a_2 - b_2 x|^\ell} \right) |\nu_1(x) - \nu_0(x)| dt.$$

Bundan, (6) ni inobatga olsak

$$\begin{aligned} |\nu_2(x) - \nu_1(x)| &\leq M \left(\frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right) \times \\ &\times \left(|\mu_1| K_1 M (1-x)^{\beta-\alpha} \int_x^1 \frac{dt}{|t - a_1 - b_1 x|^\ell} + |\mu_2| K_2 M (1-x)^{\beta-\alpha} \int_x^1 \frac{dt}{|t - a_1 - b_1 x|^\ell} \right) = \\ &= M \left(\frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right)^2 \end{aligned} \quad (8)$$

Bu jarayonni davom ettirsak:

$$|\nu_n(x) - \nu_{n-1}(x)| \leq M \left(\frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right)^n. \quad (9)$$

ga ega bo'lamiz.

Natijada quyidagi funksional qatorni tuzamiz.

$$\nu_0(x) + (\nu_1(x) - \nu_0(x)) + (\nu_2(x) - \nu_1(x)) + \dots + (\nu_n(x) - \nu_{n-1}(x)) + \dots \quad (10)$$

Bu qator

$$M \sum_{k=0}^{\infty} \left(\frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right)^k \quad (11)$$

sonli qator bilan mo'joranta bo'ladi.

Shunday qilib, funksional qator [2]

$$\left| \frac{|\mu_1| K_1(a_1^{1-\ell} + b_1^{1-\ell}) + |\mu_2| K_2(a_2^{1-\ell} + b_2^{1-\ell})}{1 - \ell} \right| < 1 \quad (12)$$

shart bajarilganida tekis yaqinlashadi va uning yig'indisi uzluksiz funksiyadir [4].

Aytib o'tish kerakki, bir jinsli (1) integral tenglama notrivial yechimga ega. Haqiqatan ham, ushbu bir jinsli tenglamani qaraylik

$$\nu(x) = \sum_{k=1}^2 \int_x^1 \frac{K_k(x, t) \nu(t) dt}{|t - a_k - b_k x|^{1-\alpha+\beta}} \quad (13)$$

yoki

$$\begin{aligned} \nu(x) &= \sum_{k=1}^2 \int_x^{a_k+b_k x} \frac{\mu_k \beta (1-\alpha) (1-x)^\beta b_k F(\alpha, 1-\beta, 2; b_k(t-x)/a_k(1-t)) \nu(t) dt}{a_k^\alpha (1-t)^\alpha (a_k + b_k x - t)^{1-\alpha+\beta}} + \\ &+ \sum_{k=1}^2 \int_{a_k+b_k x}^1 \frac{\mu_k \Gamma(1-\beta) (1-\alpha) a_k^{1-\alpha-\beta} (1-t)^{1-\alpha-\beta} (1-x)^\beta}{b_k^{1-\beta} (t-x)^{1-\beta} \Gamma(2-\alpha-\beta) \Gamma(\alpha) (t-a_k-b_k)^{1-\alpha+\beta}} \times \end{aligned} \quad (14)$$

$$\times F(-\beta, 1-\beta, 2-\alpha-\beta; a_k(1-t)/b_k(t-x))v(t)dt.$$

(14) ning o'ng tomonidagi birinchi yig'indida $t = x + (a_k + b_k x - x)\sigma = x + a_k(1-x)\sigma$ almashtirishni, ikkinchi yig'indiga esa $t = a_k + b_k x + (1-a_k - b_k x)\sigma = a_k + b_k x + b_k(1-x)\sigma$ almashtirishni bajaramiz.

Bunday almashtirish yordamida (14) tenglamani quyidagi ko'rinishga olib kelamiz

$$\begin{aligned} v(x) = & \sum_{k=1}^2 \int_0^1 \frac{\mu_k \beta (1-\alpha) F(\alpha, 1-\beta, 2; b_k \sigma / (1-a_k \sigma))}{a_k^\beta (1-a_k \sigma)^\alpha (1-\sigma)^{1-\alpha+\beta}} v[x + a_k(1-x)\sigma] d\sigma + \\ & + \sum_{k=1}^2 \int_0^1 \frac{\mu_k \Gamma(1-\beta)(1-\alpha) a_k^{1-\alpha-\beta} b_k^{-\beta} v[a_k + b_k x + b_k(1-x)\sigma] d\sigma}{\Gamma(\alpha) \Gamma(2-\alpha-\beta) \sigma^{1-\alpha+\beta} (a_k + b_k \sigma)^{1-\beta}} \times \\ & \times F(-\beta, 1-\beta, 2-\alpha-\beta; (1-\sigma)/(a_k + b_k \sigma)) d\sigma. \end{aligned} \quad (15)$$

(15) munosabatdan ko'rinib turibdiki, $v(x) = (1-x)^\lambda$, funksiya agarda $\lambda > 0$ bo'lsa, μ_k larning yetarlicha kichik qiymatlarida (15) tenglamaning notrivial yechimidir [5].

Haqiqatan ham:

$$\begin{aligned} v[x + a_k(1-x)\sigma] &= (1-x-a_k(1-x)\sigma)^\lambda = (1-x)^\lambda (1-a_k \sigma)^\lambda \\ v[a_k + b_k x + b_k(1-x)\sigma] &= (b_k(1-x)(1-\sigma))^\lambda. \end{aligned}$$

Bu holda (15) munosabatdan [3,4,5]

$$\begin{aligned} (1-x)^\lambda = & \sum_{k=1}^2 \int_0^1 \frac{\mu_k \beta (1-\alpha) F(\alpha, 1-\beta, 2; b_k \sigma / (1-a_k \sigma))}{a_k^\beta (1-a_k \sigma)^\alpha (1-\sigma)^{1-\alpha+\beta}} \times \\ & \times (1-x)^\lambda (1-a_k \sigma)^\lambda d\sigma + \\ & + \sum_{k=1}^2 \int_0^1 \frac{\mu_k \Gamma(1-\beta)(1-\alpha) a_k^{1-\alpha-\beta} b_k^{-\beta} b_k^\lambda (1-x)^\lambda (1-\sigma)^\lambda}{\Gamma(\alpha) \Gamma(2-\alpha-\beta) \sigma^{1-\alpha+\beta} (a_k + b_k \sigma)^{1-\beta}} \times \\ & \times F(-\beta, 1-\beta, 2-\alpha-\beta; (1-\sigma)/(a_k + b_k \sigma)) d\sigma. \end{aligned} \quad (16)$$

kelib chiqadi.

Shunday uchun, har qanday $\lambda > 0$ ni [6] (16) tenglik bajarilishi uchun μ_1 va μ_2 tanlash mumkin.

Bundan kelib chiqadiki, (16) tenglamaning yagona yechilishi uchun (12) shart zarur [6].

Xulosa. Tadqiqot natijalari shuni ko'rsatadiki, Volterra tipidagi integral tenglamalarda siljishli nuqtalarni hisobga olish yechimlarning aniqligi va barqarorligini oshirishga yordam beradi. Shu bilan birga, maqola kelajakdagi tadqiqotlar uchun sonli usullarni yanada takomillashtirish va ko'proq siljishli nuqtalarni o'z ichiga olgan integral tenglamalarni o'rganish istiqbollari belgilaydi. Umuman olganda, ushbu tadqiqot Volterra tipidagi integral tenglamalarni chuqur tushunishga, ularning maxsus nuqtalar bilan bog'liq xususiyatlarini aniqlashga va amaliy masalalarda qo'llash imkoniyatlarini kengaytirishga xizmat qiladi.

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KUTISH JOYLARI SONI CHEKLI BO'LGAN XIZMAT KO'RSATISH SISTEMALARI
STATSIONAR NAVBAT UZUNLIKLARI O'RTASIDAGI MUNOSABATLAR

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Annotatsiya. Ushbu maqolada kutish joylari soni cheklangan va kutish joylari soni cheklanmagan xizmat ko'rsatish sistemalari statsionar navbat uzunliklari taqsimotlari o'rtasidagi ayrim munosabatlar tahlil qilinadi. Ma'lum shartlarda $M|G|1$ va $M|G|1|N$ sistemalar, shuningdek, $\bar{M}_2|\bar{G}_2|I|N_1N_2$ va $\bar{M}_2|\bar{G}_2|I|N_1, \infty$ imtiyozli sistemalarni oddiy talablar navbat uzunluklari taqsimotlari o'rtasida aloqa o'rnatish mumkinligi ko'rsatiladi. O'rnatilgan munosabatlar bir sistema uchun olingan qator natijalarini ikkinchi sistemaga o'tkazish, shu bilan birga yuklanishi, imtiyozli sistema yangi ilmiy natijalar olish imkonini beradi. Ishda olingan natijalarning ayrim tatbiqlari ham qaralgan.

Kalit so'zlar: xizmat ko'rsatish sistemasi, bandlik davri, navbat uzunligi, sistemaning yuklanishi, imtiyozli sistema.

СООТНОШЕНИЯ МЕЖДУ СТАЦИОНАРНЫМИ ДЛИНАМИ ОЧЕРЕДЕЙ В
СИСТЕМАХ МАССОВОГО ОБСЛУЖИВАНИЯ С ОГРАНИЧЕННЫМ ЧИСЛОМ МЕСТ ДЛЯ
ОЖИДАНИЯ

Аннотация. В данной статье анализируются некоторые взаимосвязи между стационарными распределениями длины очереди в системах массового обслуживания с ограниченным и неограниченным числом мест для ожидания. Показано, что при определенных условиях можно установить связь между распределениями длины очереди обычных требований в системах $M|G|1$ и $M|G|1|N$, а также в приоритетных системах типа $\bar{M}_2|\bar{G}_2|I|N_1N_2$ и $\bar{M}_2|\bar{G}_2|I|N_1, \infty$. Установленные соотношения позволяют переносить ряд результатов, полученных для одной системы, на другую, что открывает возможности для получения новых научных выводов в области теории массового обслуживания и анализа приоритетных систем при различных уровнях нагрузки. В работе также рассмотрены некоторые практические приложения полученных результатов.

Ключевые слова: система массового обслуживания, период занятости, длина очереди, загрузка системы, приоритетная система.

SERVICE SYSTEMS WHERE THE NUMBER OF WAITING AREAS IS FINITE ARE THE
RELATIONSHIPS BETWEEN STATIONARY QUEUE LENGTHS

Abstract. This article analyzes certain relationships between the stationary queue length distributions of service systems with limited and unlimited waiting spaces. It is demonstrated that under specific conditions, connections can be established between the queue length distributions of ordinary requirements for $M|G|1$ and $M|G|1|N$ systems, as well as for $\bar{M}_2|\bar{G}_2|I|N_1N_2$ and $\bar{M}_2|\bar{G}_2|I|N_1, \infty$ infty priority systems. The established relationships allow for the transfer of results obtained for one system to another, thereby enabling the generation of new scientific results. Furthermore, some practical applications of the results obtained in this work are considered.

Keywords: Queueing system, busy period, queue length, system workload, priority system.

Kirish. Ma'lumki, xizmat ko'rsatilishi lozim bo'lgan talablar xizmat obyektlarida (sanoat korxonalari, savdo tarmoqlari, transport, aloqa va boshqa kommunikatsiya tuzilmalari va h.k) to'planib qolib, uzun navbatlar hosil bo'ladigan bo'lsa, bu nafaqat vaqt yo'qotish, balki resurslarning vaqtida ishlatilmasligi, ularni

saqlash, mahsulotlarning eskirishi, ma'lumotlarning vaqtida uzatilmaligi natijasida kelib chiqadigan katta miqdordagi moddiy zarar muammolarini keltirib chiqaradi. Shu sababli xizmat ko'rsatish korxonalarini to'g'ri rejalashtirish, samarali xizmat ko'rsatishni ta'minlash muhim hayotiy ahamiyatga ega. Xizmat jarayoni samaradorligini baholash uchun, avvalo, sistemaning asosiy xarakteristikalaridan bo'lgan navbat uzunligining taqsimoti va sonli xarakteristikalarini bilish talab etiladi.

$M|M|m$ tipidagi klassik hamda $M|G|m$ tipdagi real holatga yaqin sistemalar yaxshi o'rganilgan va bu bo'yicha yetarli adabiyotlar (manografiya, darslik, ilmiy maqolalar) mavjud. Lekin $M|G|m|N$ (kutish joylari soni chekli) ko'rinishdagi sistemalar ma'lum qiyinchiliklar sababli yetarlicha to'liq o'rganilmagan. [3] ishda $M|G|1|N$ sistemaning statsionar, [4] ishda nostatsionar navbat uzunligi taqsimotini aniqlovchi munosabatlar keltirib chiqarilgan bo'lsada, ushbu natijalar hosil qiluvchi funksiyalar va Laplas-Stiltes almashtirishlari orqali ifodalangan. Bu esa amalda ushbu natijalardan foydalanishda qiyinchilik tug'diradi. [5] ishda $M|G|1|N$ va $GM|G|1|N$ - sistemalar statsionar navbat uzunliklari taqsimotlarining bandlik davri o'rtacha qiymatlari orqali ifodalangan ko'rinishi topilgan. Ma'lumki, sonli xarakteristikalar nisbatan oson tahlil qilinadi.

Qaralayotgan ishda ma'lum shartlarda $M|G|1|N$ sistema statsionar navbat uzunligi taqsimotining yetarlicha yaxshi o'rganilgan $M|G|1$ sistema statsionar navbat uzunligi taqsimoti orqali ifodalangan ko'rinishi aniqlanadi. Shu bilan bir qatorda shu tipdagi munosabat $\bar{M}_2|\bar{G}_2|1|N_1N_2$ va $\bar{M}_2|\bar{G}_2|1|N_1, \infty$ imtiyozli sistemalarning oddiy talablar navbat uzunliklari taqsimotlari o'rtasida ham o'rnatiladi. Ishda o'rnatilgan munosabatlar tatbiqi sifatida $M|G|1|N$ sistema statsionar navbat uzunligi taqsimotining asimtotik ko'rinishlari keltirib chiqariladi.

1. $M|G|1$ va $M|G|1|N$ sistemalar statsionar navbat uzunliklari taqsimotlari o'rtasida o'zaro munosabatlar

Bir kanalli xizmat ko'rsatish sistemasiga λ parametrli Puasson talablar oqimi kelib tushayotgan bo'lsin, ya'ni talablar kelib tushgan qo'shni momentlar orasidagi vaqt uzunliklari o'zaro bog'liq emas va bir xil

$$A(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0, \lambda > 0 \end{cases}$$

taqsimot funksiyasiga ega. Talablarga ularning kelish tartibida xizmat ko'rsatiladi va xizmat vaqti uzunliklari o'zaro bog'liq bo'lmagan hamda bir xil $B(x)$ [$B(+0)=0$] taqsimotga va μ^{-1} o'rtacha qiymatga ega bo'lgan tasodifiy miqdorlarni tashkil etadi, deb hisoblaymiz. Kutish joylari soni N ($N \geq 1$) ga teng, ya'ni sistemada (xizmatdagi talab bilan birga) ko'pi bilan $N+1$ ta talab bo'lishi mumkin. Agar talab kelgan momentda hamma o'rinlar band bo'lsa, u sistemaga qabul qilinmaydi va keyingi xizmat jarayoniga hech qanday ta'sir ko'rsatmaydi. Yuqorida ta'riflangan sistema Kendallga ko'ra $M|G|1|N$ deb belgilanadi.

Agar $N=\infty$ bo'lsa, $M|G|1|\infty$ kutishli sistema deyiladi. Bu holda odatda ∞ yozilmaydi va sistema $M|G|1$ deb belgilanadi.

Quyidagi belgilashlarni kiritamiz:

ξ_N va ξ – mos holda $M|G|1|N$ va $M|G|1$ sistemalarning statsionar (vaqtga bog'liq bo'lmagan) navbat uzunliklari:

ζ_k - $M|G|1|k$, $k = \overline{1, N}$, sistemaning bandlik davri;

$\rho = \lambda \mu^{-1}$ - sistemalarning yuklanishi.

$P_N(k) = P(\xi_N = k)$, $\overline{0, N+1}$,

$P(k) = P(\xi = k)$, $k \geq 0$.

1-teorema. $\rho < 1$ da quyidagi munosabat o'rinli:

$$P_N(k) = \frac{P(k)}{1 - \rho + \rho P(\xi \leq N)}, \quad k = \overline{0, N}, \quad (1)$$

$$P_N(N+1) = \frac{(1-\rho)(1-P(\xi \leq N))}{1 - \rho + \rho P(\xi \leq N)}, \quad (1')$$

Isbot. [5] ishda ushbu munosabatlar keltirilgan:

$$P_N(0) = \frac{1}{1 + \lambda M \zeta_N},$$

$$P_N(k) = \frac{\mu(M \zeta_k - M \zeta_{k-1})}{1 + \lambda_2 M \zeta_N}, \quad k = \overline{1, N}, \quad (2)$$

$$P_N(N+1) = \frac{1 - (\mu - \lambda) M \zeta_N}{1 + \lambda M \zeta_N},$$

Ma'lumki, ([1], 195 – bet), $\rho < 1$ da

$$\lim_{N \rightarrow \infty} M \zeta_N = \frac{1}{\mu(1-\rho)}.$$

Bu munosabatlarga ko'ra (2) dan hosil qilamiz.

$$P(0) = 1 - \rho.$$

$$P(k) = \mu(1-\rho)(M\zeta_k - M\zeta_{k-1}), k \geq 1 \quad (3)$$

(3) ga ko'ra (2) dan

$$P_N(k) = \frac{P(k)}{(1-\rho)(1+\lambda M \zeta_N)} \quad (4)$$

tenglamaga ega bo'lamiz. (3) munosabatga ko'ra

$$P(\xi \leq N) = \mu(1-\rho)M\zeta_k$$

yoki

$$M\zeta = \frac{P(\xi \leq N)}{\mu(1-\rho)}$$

tenglikni va bunga ko'ra

$$(1-\rho)(1+\lambda M \zeta_N) = (1-\rho) \left(1 + \frac{\rho P(\xi \leq N)}{1-\rho} \right) = 1-\rho + \rho P(\xi \leq N)$$

munosabatni hosil qilamiz. Ushbu tenglikga ko'ra (4) dan (1) kelib chiqadi.

Agar

$$\sum_{k=0}^{N+1} P(\xi_N = k) = 1$$

munosabatni e'tiborga olsak, (1') tenglik (1) dan bevosita hosil bo'ladi. Haqiqatdan ham yuqoridagi tenglikdan ushbu munosabatga ega bo'lamiz:

$$\begin{aligned} P(\xi_N = N+1) &= 1 - P(\xi_N \leq N) = 1 - \frac{\sum_{k=0}^N P(\xi = k)}{1-\rho + \rho P(\xi \leq N)} \\ &= 1 - \frac{P(\xi \leq N)}{1-\rho + \rho P(\xi \leq N)} = \frac{(1-\rho)(1 - P(\xi_N \leq N))}{1-\rho + \rho P(\xi \leq N)}. \end{aligned}$$

Misol. Ma'lumki, xususiyl holda, $B(x) = 1 - e^{-\mu x}$, $x > 0$ bo'lganda, $\rho < 1$ da

$$P(k) = (1-\rho)\rho^k, k \geq 0$$

formula o'rinli. Bunga ko'ra (1) dan ushbu tenglikni hosil qilamiz:

$$P_N(k) = \frac{(1-\rho)\rho^k}{1-\rho^{N+2}}$$

Ushbu formula [1] ishda (49-bet) boshqacha usulda keltirib chiqarilgan.

Tenglikning tatbiqi sifatida quyidagi teoremani isbot qilamiz.

2-teorema. Agar $\sigma^2 = \int_0^\infty x^2 dB(x) < \infty$ va $N \rightarrow \infty$, $\rho \uparrow 1$ da

$N(1-\rho) \rightarrow \alpha$ ($0 \leq \alpha \leq \infty$) bo'lsa, u holda ushbu asimtotik munosabatlar o'rinli:

$$\lim P\left(\xi_N \frac{1-\rho}{1-\rho^N} < x\right) = \begin{cases} x, & \alpha = 0, \\ 1 - e^{-\frac{mx}{A}}, & 0 < \alpha < \infty, \\ 1 - e^{-mx}, & \alpha = \infty. \end{cases} \quad (5)$$

bu yerda $m = \frac{2}{\mu^2 \sigma^2}$, $A = \alpha(1 - e^{-\alpha})^{-1}$.

Isbot. $N \rightarrow \infty$ da $\rho \uparrow 1$ da $N(1-\rho) \rightarrow \alpha$ ($0 \leq \alpha \leq \infty$) bo'lsin.

[2] (171-bet) ishda quyidagi munosabat keltirilgan: agar $\sigma^2 < \infty$ bo'lsa, u holda

$$\lim P(\xi(1-\rho) < x) = 1 - e^{-mx} \quad (6)$$

(1) tenglikga ko'ra

$$P\left(\xi_N \frac{1-\rho}{1-\rho^N} < x\right) = \frac{P\left(\xi \frac{1-\rho}{1-\rho^N} < x\right)}{1-\rho + \rho P(\xi \leq N)} \quad (7)$$

Quyidagi munosabat o'rinli: $N(1-\rho) \rightarrow \alpha$ ($0 \leq \alpha \leq \infty$) da

$$\rho^N = \left(1 - \frac{(1-\rho)N}{N}\right)^N \rightarrow e^{-\alpha} \quad (8)$$

va (6) ga ko'ra

$$P(\xi \leq N) = P(\xi(1-\rho) \leq N(1-\rho)) \rightarrow 1 - e^{-m\alpha x}. \quad (9)$$

(7) dan (8) va (9) larga ko'ra ushbu tenglikni hosil qilamiz:

$$P\left(\xi_N \frac{1-\rho}{1-\rho^N} < x\right) = \frac{1-e^{-m(1-e^{-\alpha})x}}{1-\rho+\rho P(\xi \leq N)} \quad (10)$$

Bu yerda $A = \alpha(1 - e^{-\alpha})^{-1}$ belgilash kiritib, (5) munosabatning

$0 < \alpha < \infty$ dagi ko'rinishiga ega bo'lamiz. Agar (10) dagi limitik taqsimot

α bo'yicha uzluksizligi e'tiborga olinsa, $\alpha = 0$ va $\alpha = \infty$ bo'lgan hollarni (10) tenglikdan

bevosita $\alpha \rightarrow 0$ va $\alpha \rightarrow \infty$ da limitga o'tib hosil qilish mumkin.

2. $\vec{M}_2|\vec{G}_2|1|N_1, \infty$ va $\vec{M}_2|\vec{G}_2|1|N_1N_2$ imtiyozli xizmat ko'rsatish sistemalarida oddiy talablar navbat uzunliklari o'rtasidagi munosabat

Quyidagi shartlar bilan berilgan $F(N_1) - \vec{M}_2|\vec{G}_2|1|\vec{N}_1, \infty$ va $F(N_1, N_2) - \vec{M}_2|\vec{G}_2|1|N_1N_2$ bir kanalli xizmat ko'rsatish sistemalari qaralayotgan bo'lsin:

Sistemaga mos holda λ_1 va λ_2 parametrlri Puasson talablar oqimini tashkil etuvchi shoshilinch va oddiy talablar kelib tushadi. Shoshilinch va oddiy talablarga xizmat ko'rsatish vaqti uzunliklari o'zaro bo'g'liq bo'lmagan va mos holda μ_1 va μ_2 parametrlri ko'rsatkichli taqsimotga ega bo'lgan tasodifiy miqdorlarni tashkil etadi. Shoshilinch va oddiy talablarning kutish joylari soni mos holda N_1 ($N_1 \geq 1$) va N_2 ($N_2 \geq 1$) ga teng. Xizmat ko'rsatish tartibi: oxirigacha xizmat ko'rsatiladigan mutlaq imtiyozdan iborat.

Quyidagi belgilashlarni kiritamiz:

$\zeta_{N_1k} - F(N_1, k), k \geq 1$, sistemaning bandlik davri;

ζ_{N_1} - sistemaning shoshilinch talablar bilan bandlik davri;

$\xi_{N_1N_2}$ va ξ_{N_1} - mos holda $F(N_1, N_2)$ va $F(N_1)$ sistemalarni statsionar navbat uzunliklari;

$\rho_i = \lambda_i \mu_i^{-1}, \quad i = 1, 2;$

$$\rho = \rho_2(1 + \lambda_1 M \zeta_{N_1})$$

Ushbu ishda $\rho < 1$ da $\xi_{N_1N_2}$ va ξ_{N_1} tasodifiy miqdorlarning taqsimotlari o'rtasida bo'g'liqlik o'rnatiladi.

3 - teorema. $\rho < 1$ da quyidagi munosabat o'rinli,

$$P(\xi_{N_1N_2} = k) = \frac{(1-\rho)P(\xi_{N_1} = k)}{1-\rho+\rho_1(1+\rho_2-\rho)P(\xi_{N_1} \leq N_2)}, \quad 0 \leq k \leq N_2,$$

$$P(\xi_{N_1N_2} = N_2 + 1) = \frac{1-\rho+\rho_2(1+\rho_2-\rho)P(\xi_{N_1} \leq N_2)}{1-\rho+(\rho_2^2-(1-\rho)P(\xi_{N_1} \leq N_2))},$$

Izoh. Teoremada keltirilgan tengliklarga o'xshash munosabatlar $M|G|1$ va $M|G|1|N$ sistemalar statsionar navbat uzunliklari taqsimotlari uchun 1- teoremada o'rnatilgan.

3-teoremaning isboti. Qaralayotgan imtiyozli sistemalarni faqat oddiy talablarga xizmat ko'rsatayotgan va xizmat vaqti uzunligi xizmat ko'rsatish sikliga teng bo'lgan oddiy sistemaga keltirish mumkin.

Xizmat ko'rsatish sikli deb, oddiy talabning qurilmaga kelib tushish momentidan uning sistemadan chiqib ketish momentigacha bo'lgan vaqt uzunligiga aytiladi. Ushbu belgilanishlarni kiritaylik:

C-oddiy talabga xizmat ko'rsatish sikli;

$$C(s) = b(s + \lambda_1 - \lambda_1 g_{N_1}(s)) \quad (11)$$

$b(s) = \int_0^\infty e^{-sx} dB(x)$, $B(x)$ - oddiy talabga xizmat ko'rsatish vaqti, $Res \geq 0$;

$$g_{N_1}(s) = \int_0^\infty e^{-sx} P(\xi_{N_1} < x); Res \geq 0;$$

(11) munosabat [6] ishda $N_1 = \infty$ uchun keltirib chiqarilgan tenglikning isbotiga o'xshash isbot qilinadi. Shu sababli uning isbotini keltirib o'tirmaymiz.

(11) munosabatdan ushbu tenglikka ega bo'lamiz:

$$MC = \frac{1}{\mu^2} (1 + \lambda_1 M \xi_{N_1}),$$

U holda xizmat davomiyligi xizmat ko'rsatish sikliga teng bo'lgan sistemaning yuklanishi quyidagiga teng bo'ladi:

$$\rho = \lambda_2 MC = \rho_2(1 + \lambda_1 M \xi_{N_1}) \quad (12)$$

Bundan keyin sistemaga faqat oddiy talablar kelib tushyapti va ularning xizmat vaqti uzunligi xizmat ko'rsatish sikliga teng deb hisoblaymiz.

[5] ishda quyidagi munosabatlar o'rnatilgan edi:

$$P(\xi_{N_1 N_2} = 0) = \frac{1}{1 + \lambda_2 M \zeta_{N_1 N_2}}, \quad (13)$$

$$P(\xi_{N_1 N_2} = k) = \frac{\mu_2 (M \zeta_{N_1 k} - M \zeta_{N_1 k-1})}{1 + \lambda_2 M \zeta_{N_1 N_2}}, \quad 1 \leq k \leq N_2,$$

Ma'lumki, $\rho < 1$ da ([2], 62 – bet)

$$\lim_{N_2 \rightarrow \infty} M \zeta_{N_1 N_2} = M \zeta_{N_1 \rightarrow \infty} = \frac{1}{\mu_2 (1 - \rho)}.$$

Ushbu munosabatni e'tiborga olib, (13) dan $N_2 \rightarrow \infty$ da quyidagi tengliklarga ega bo'lamiz:

$$P(\xi_{N_1 N_2} = 0) \rightarrow P(\xi_{N_1} = 0) = \frac{1 - \rho}{1 - \rho + \rho_2}$$

$$P(\xi_{N_1 N_2} = k) \rightarrow \frac{\mu_2 (1 - \rho)(M \zeta_{N_1 k} - M \zeta_{N_1 k-1})}{1 - \rho + \rho_2}, \quad 1 \leq k \leq N_2 \quad (14)$$

(13) va (14) ga ko'ra

$$P(\xi_{N_1 N_2} = k) = \frac{P(\xi_{N_1} = k)(1 - \rho + \rho_2)}{(1 - \rho)(1 + \lambda_2 M \zeta_{N_1 N_2})}, \quad (15)$$

munosabatlarni hosil qilamiz. Agar (14) ga asosan

$$P(\xi_{N_1} \leq N_2) = \sum_{k=0}^{N_2} P(\xi_{N_1} = k) \frac{\mu_2 (1 - \rho)}{1 - \rho + \rho_2} M \zeta_{N_1 N_2}$$

tenglik o'rinli bo'lishini etiborga olsak, u xolda (15) ushbu ko'rinishga keladi.

$$P(\xi_{N_1 N_2} = k) = P(\xi_{N_1} = k) \left(1 + \frac{\rho_2 (1 - \rho + \rho_2)}{(1 - \rho)} P(\xi_{N_1} \leq N_2) \right)^{-1} \quad (16)$$

Agar

$$P(\xi_{N_1 N_2} = N_2 + 1) = 1 - \sum_{k=0}^{N_2} P(\xi_{N_1 N_2} = k)$$

munosabat e'tiborga olinsa, (16) dan teoremda keltirilgan ikkinchi tenglik kelib chiqadi.

Xulosa. Shuni qayd etib o'tamizki, agar statsionar rejim mavjud bo'lsa, ommaviy xizmat ko'rsatish sistemalarining ko'plab modellarida kutishli va navbat uzunligi chegaralaridan sistemalar navbat uzunliklari o'rtasida o'zaro munosabatlar o'rnatish mumkin. Ushbu ishda keltirilgan natijalar ma'lum shartlarda statsionar kutish vaqtlari taqsimotlari uchun ham o'rinli ekanligini ko'rsatish mumkin. Ma'lumki, kutish vaqti taqsimotini tahlil qilish vaqt yo'qotish bilan bog'liq muammolarini hal etishda juda muhim ahamiyatga ega.

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ON THE MECHANISM OF EIGENVALUE CREATION AND ANNIHILATION FOR THE ONE-DIMENSIONAL LATTICE SCHRÖDINGER OPERATOR

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Abstract. This work examines the spectral properties of a one-dimensional lattice Schrödinger operator $H_{\mu\lambda}$ with two interaction constants. We establish the precise conditions under which the number of isolated eigenvalues below the essential spectrum changes. Utilizing the Fredholm determinant method, we partition the (μ, λ) parameter space into stability regions with a constant number of eigenvalues ($k = 0, 1, 2$). Explicit algebraic equations for the boundaries of these regions are obtained. We provide a complete characterization of the eigenvalue transition mechanism at the edge of the essential spectrum ($\mathcal{E}_{\min} = 0$), identifying the exact coupling constants that trigger spectral phase transitions.

Keywords: Two-particle system, discrete Schrödinger operator, hamiltonian, Lippmann-Schwinger Operator, essential spectrum, Fredholm determinant, eigenvalue, critical operator.

О МЕХАНИЗМЕ ВОЗНИКНОВЕНИЯ И ИСЧЕЗНОВЕНИЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ ОДНОМЕРНОГО РЕШЁТЧАТОГО ОПЕРАТОРА ШРЁДИНГЕРА

Аннотация. В работе изучаются спектральные свойства одномерного решётчатого оператора Шрёдингера $H_{\mu\lambda}$ с двумя константами взаимодействия. Установлены точные условия изменения числа собственных значений ниже существенного спектра. Методом определителя Фредгольма пространство параметров (μ, λ) разбито на области с постоянным числом собственных значений ($k=0, 1, 2$). Получены явные уравнения границ этих областей. Описан механизм перехода собственных значений на краю спектра ($\mathcal{E}_{\min} = 0$) и найдены значения параметров, при которых происходят спектральные фазовые переходы.

Ключевые слова: Двухчастичная система, дискретный оператор Шрёдингера, гамильтониан, оператор Липпмана-Швингера, существенный спектр, определитель Фредгольма, собственное значение, критический оператор.

BIR O'LCHAMLI PANJARADA SHRODINGER OPERATORI UCHUN XOS QIYMATLARNING HOSIL BO'LISH VA YO'QOLISH MEXANIZMI

Annotatsiya. Ushbu ishda ikkita o'zaro ta'sir konstantasiga ega $H_{\mu\lambda}$ bir o'lchamli panjarali

Shrodinger operatorining spektral xossalari o'rganiladi. Muhim spektrdan pastdagi xos qiymatlar soni o'zgarishining aniq shartlari topildi. Fredgolm determinanti usuli bilan (μ, λ) parametrlar sohasi xos

qiymatlar soni o'zgarish ($k=0, 1, 2$) bo'lgan qismlarga ajratildi. Ushbu sohalar chegaralari uchun algebraik tenglamalar olindi. Muhim spektr chegarasida ($\mathcal{E}_{\min} = 0$) xos qiymatlarning o'tish mexanizmi ko'rsatildi va

spektral o'tishlar sodir bo'ladigan parametr qiymatlari aniqlandi.

Kalit so'zlar: Ikki zarrali sistema, diskret Shredinger operatori, gamiltonian, Lippman-Shvinger operatori, muhim spektr, Fredgolm determinanti, xos qiymat, kritik operator.

Introduction. The spectral analysis of discrete Schrödinger operators is a well-established theme in mathematical physics and operator theory. In translation-invariant lattice models, the interplay between dispersion and interaction produces a rich threshold structure that differs substantially from the continuum case. In particular, the boundedness of the lattice dispersion relation confines the essential spectrum to a compact interval and allows discrete eigenvalues to appear at both spectral edges. This feature gives rise to phenomena with no direct continuum analogue, including bound states above the essential spectrum.

The rigorous framework for few-particle Hamiltonians on lattices was developed in the works stimulated by the pioneering physics perspective of Mattis [8] and has since become an active area of mathematical physics; see, for example, [5,7,6,12,13,2,14,3,4].

Lattice Schrödinger operators also arise naturally in the theory of ultracold atoms in optical lattices [9]. In such systems, repulsively bound states have been predicted and observed experimentally [10, 11]. From a mathematical point of view, these models provide a natural setting in which the bounded periodic dispersion leads to nontrivial spectral effects at both lower and upper thresholds. This makes the lattice case especially suitable for an analysis based on Fredholm determinants, threshold expansions, and the geometry of the corresponding critical parameter sets.

The one-particle 1D lattice Schrödinger operator is typically addressed via the general perturbation theory applied to infinite Jacobi matrices (see, e.g., [1, 15]). However, introducing explicit interaction potentials governed by specific coupling constants allows for a detailed analysis of spectral shifts and the existence of bound states (isolated eigenvalues). Such models are relevant for describing single particles traveling through periodic structures, such as cold atoms in optical lattices [9, 10].

In this work, we investigate the spectral properties of a family of Hamiltonians describing a single particle on a one-dimensional lattice $\mathbb{Z}$ subject to extended interactions. The system is governed by the Schrödinger operator $H_{\mu\lambda}$. Our primary focus is to clarify the mechanism by which the number of isolated eigenvalues varies as the coupling constants are tuned. In the momentum representation, the fiber Schrödinger operator $H_{\mu\lambda}$ takes the form:

$$H_{\mu\lambda} = H_0 + V_{\mu\lambda}, \quad \mu, \lambda \in \mathbb{R}.$$

The unperturbed operator H_0 is defined as the multiplication operator by the single-particle dispersion function $\varepsilon(p) = 2(1 - \cos p)$:

$$H_0 f(p) = \varepsilon(p) f(p), \quad p \in \mathbb{T} = (-\pi, \pi].$$

The interaction term $V_{\mu\lambda}$ is an integral operator with a finite-rank kernel $v_{\mu\lambda}(p)$ that accounts for on-site and nearest-neighbor interactions:

$$v_{\mu\lambda}(p) = \mu + 2\lambda \cos p, \quad p \in \mathbb{T}.$$

Given that H_0 and $V_{\mu\lambda}$ are bounded and self-adjoint, the essential spectrum of the total Hamiltonian $H_{\mu\lambda}$ coincides with that of the free operator: $\sigma_{\text{ess}}(H_{\mu\lambda}) = \sigma_{\text{ess}}(H_0) = [\varepsilon_{\min}, \varepsilon_{\max}]$, where $\varepsilon_{\min} = 0$ and $\varepsilon_{\max} = 4$.

By employing the Lippmann-Schwinger approach and analyzing the zeros of the associated Fredholm determinant $\Delta_{\mu\lambda}(z)$, we establish a rigorous correspondence between these zeros and the isolated eigenvalues of $H_{\mu\lambda}$. Our analysis reveals that the two-dimensional parameter space (μ, λ) is precisely partitioned by algebraic curves into distinct spectral regions.

Specifically, by determining the asymptotic expansions of $\Delta_{\mu\lambda}(z)$ near the threshold of the essential spectrum $(\varepsilon_{\min} = 0)$, we provide a complete characterization of the discrete spectrum. This allows us to identify the exact thresholds at which eigenvalues emerge from or vanish into the continuous part of the spectrum.

(1) We explicitly define the curves ℓ_0, ℓ_1 that separate regions $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$, with a different number of isolated eigenvalues $(k = 0, 1, 2, \dots)$.

(2) We fully characterize the spectral phase transitions (eigenvalue birth and annihilation) in the region $(-\infty, 0)$. These transitions can appear or vanish only along the specific critical surfaces, providing the necessary and sufficient explicit conditions for the spectral change.

Initial spectral analyses of discrete Schrödinger operators, focusing on the existence and position of eigenvalues as functions of interaction parameters, were previously performed in related contexts [17, 16]. Our current research deepens this understanding by focusing specifically on the change mechanism in the simplest lattice system with two coupling constants.

The structure of this paper is as follows: Section 2 provides a detailed introduction to the single-particle lattice Schrödinger operator $H_{\mu\lambda}$ and its fundamental properties. In Section 3, we state our primary spectral results, including the partitioning of the parameter space. Finally, Section 3 is devoted to the rigorous proofs of these results.

The schrodinger operator with rank-two perturbation. In the Hilbert space $L^2_{\mathbb{R}}(\mathbb{T})$ of even, square-integrable functions, we consider the single-particle Hamiltonian defined by the following expression:

$$H_{\mu\lambda} = H_0 + V_{\mu\lambda}.$$

The unperturbed operator H_0 is the multiplication operator by the dispersion function:

$$(H_0 f)(p) = \varepsilon(p) f(p), \varepsilon(p) = 2(1 - \cos p),$$

which represents the kinetic energy of the particle on the lattice. The perturbation $V_{\mu\lambda}$ is a rank-two self-adjoint operator acting as:

$$(V_{\mu\lambda} f)(p) = \mu(f, \alpha_1) \alpha_1(p) + \lambda(f, \alpha_2) \alpha_2(p), \quad (2.1)$$

where $(\cdot, \cdot)$ denotes the standard inner product in $L^2_{\varepsilon}(\mathbb{T})$. The functions α_1 and α_2 are orthonormal basis vectors given by:

$$\alpha_1(p) = \frac{1}{\sqrt{2\pi}}, \quad \alpha_2(p) = \frac{\cos p}{\sqrt{\pi}}$$

2.1. Essential Spectrum. The spectrum of the free Hamiltonian H_0 consists of a single energy band $\sigma(H_0) = [0, 4]$. Since the perturbation $V_{\mu\lambda}$ is a finite-rank operator ($\text{rank}(V_{\mu\lambda}) \leq 2$), Weyl's theorem ensures the invariance of the essential spectrum:

$$\sigma_{\text{ess}}(H_{\mu\lambda}) = \sigma(H_0) = [0, 4]$$

As $H_{\mu\lambda}$ is self-adjoint, its discrete spectrum $\sigma_{\text{disc}}(H_{\mu\lambda})$ consists of real isolated eigenvalues of finite multiplicity, which can only accumulate at the threshold points $z = 0$ and $z = 4$. Any such eigenvalues must lie in the spectral gaps:

$$\sigma_{\text{disc}}(H_{\mu\lambda}) \subset (-\infty, 0) \cup (4, +\infty)$$

Let $n_-(H_{\mu\lambda})$ and $n_+(H_{\mu\lambda})$ denote the number of discrete eigenvalues (counting multiplicities) located below and above the essential spectrum, respectively. The finite-rank nature of $V_{\mu\lambda}$ imposes the following global bound:

$$n(H_{\mu\lambda}) := n_-(H_{\mu\lambda}) + n_+(H_{\mu\lambda}) \leq 2.$$

Remark 2.1. The distribution of eigenvalues in the spectral gaps is governed by the signs of the coupling constants μ and λ . If $\mu, \lambda \geq 0$ (resp. $\mu, \lambda \leq 0$), then $V_{\mu\lambda}$ is a positive (resp. negative) operator, and eigenvalues may only appear above (resp. below) the essential spectrum. In the indefinite case where $\mu\lambda < 0$, the operator $H_{\mu\lambda}$ can simultaneously possess eigenvalues in both gaps. In the following sections, we shall primarily focus on the emergence of the discrete spectrum in the lower gap $(-\infty, 0)$ and classify the parameter regions where $n_-(H_{\mu\lambda})$ takes the values 0, 1, or 2.

2.2. The Lippmann-Schwinger Operator. In this subsection, we reduce the eigenvalue problem for $H_{\mu\lambda}$ to a finite-dimensional algebraic problem by introducing an auxiliary finite-rank operator.

Definition 2.2. For a fixed $z \in \mathbb{R} \setminus [0, 4]$, the Lippmann-Schwinger operator $B_{\mu\lambda}(z)$ associated with the Hamiltonian $H_{\mu\lambda}$ and defined on the Hilbert space $L^2_{\varepsilon}(\mathbb{T})$ is given by the formula

$$B_{\mu\lambda}(z) = V_{\mu\lambda} R_0(z)$$

where $R_0(z) = (H_0 - zI)^{-1}$ is the free resolvent operator and $V_{\mu\lambda}$ is the rank-two perturbation defined in 2.1.

The operator $B_{\mu\lambda}(z)$ is of finite rank ($\text{rank } B_{\mu\lambda}(z) \leq 2$). This property is fundamental for reducing the spectral analysis of the infinite-dimensional operator $H_{\mu\lambda}$ to the study of the zeros of a finite-dimensional determinant.

Theorem 2.3. For any $\mu, \lambda \in \mathbb{R}$, a value $z \in \mathbb{C} \setminus [0, 4]$ is an eigenvalue of the operator $H_{\mu\lambda}$ with multiplicity m if and only if 1 is an eigenvalue of the operator $B_{\mu\lambda}(z)$ with the same multiplicity m .

Proof. For $z \notin \sigma(H_0)$, the eigenvalue equation $H_{\mu\lambda} \psi = z\psi$ is equivalent to $(H_0 - V_{\mu\lambda})\psi = z\psi$, or $(H_0 - zI)\psi = V_{\mu\lambda} \psi$. Since the free resolvent $R_0(z)$ exists for $z \in \mathbb{C} \setminus [0, 4]$, we can write $\psi = R_0(z) V_{\mu\lambda} \psi$. Let $\varphi = V_{\mu\lambda} \psi$. Applying $V_{\mu\lambda}$ to both sides of the resolvent equation yields $\varphi = V_{\mu\lambda} R_0(z) \varphi$, which is precisely $B_{\mu\lambda}(z) \varphi = \varphi$. Conversely, if φ is an eigenvector of $B_{\mu\lambda}(z)$ corresponding to the eigenvalue 1, then $\psi = R_0(z) \varphi$ is the corresponding eigenvector of $H_{\mu\lambda}$. The invertibility of $R_0(z)$ ensures

that the mapping $\psi \mapsto V_{\mu\lambda}\psi$ is an isomorphism between the respective eigenspaces, thus preserving the multiplicity m .

The rank-two structure of $V_{\mu\lambda}$ reduces the equation $B_{\mu\lambda}(z)\varphi = \varphi$ to the linear system for $c_i := (\varphi, \alpha_i), i = 1, 2$:

$$\begin{cases} (1 - \mu a_{11}(z))c_1 - \lambda a_{12}(z)c_2 = 0 \\ \mu a_{21}(z)c_1 + (1 - \lambda a_{22}(z))c_2 = 0 \end{cases}$$

where the matrix elements $a_{ij}(z)$ are given by:

$$a_{ij}(z) = (R_0(z)\alpha_j, \alpha_i) = \int_{\mathbb{T}} \frac{\alpha_i(q)\alpha_j(q)}{\varepsilon(q) - z} dq.$$

The Fredholm determinant $\Delta_{\mu\lambda}(z) = \det(I - B_{\mu\lambda}(z))$ is then explicitly:

$$\Delta_{\mu\lambda}(z) = \begin{vmatrix} 1 - \mu a_{11}(z) & -\lambda a_{12}(z) \\ \mu a_{21}(z) & 1 - \lambda a_{22}(z) \end{vmatrix}.$$

Lemma 2.4. A number $z \in \mathbb{R} \setminus [0, 4]$ is an eigenvalue of $H_{\mu\lambda}$ if and only if it is a zero of the Fredholm determinant, i.e., $\Delta_{\mu\lambda}(z) = 0$.

Proof. For $z \notin [0, 4]$, the equation $(H_{\mu\lambda} - zI)\psi = 0$ is equivalent to the integral representation

$$\psi(p) = \frac{\mu c_1 \alpha_1(p) + \lambda c_2 \alpha_2(p)}{\varepsilon(p) - z}, \quad (2.2)$$

where $c_i = (\psi, \alpha_i)$ for $i = 1, 2$. Substituting (2.2) into the definitions of c_i yields the following homogeneous linear system:

$$(1 - \mu a_{11}(z) - \lambda a_{12}(z) - \mu a_{21}(z) - \lambda a_{22}(z))(c_1 c_2) = (0 \ 0).$$

A non-trivial solution $\psi \in L^2_{\varepsilon}(\mathbb{T})$ exists if and only if the system admits a non-zero solution (c_1, c_2) , which is equivalent to the vanishing of the determinant $\Delta_{\mu\lambda}(z) = \det(I - B_{\mu\lambda}(z)) = 0$. $\square$

The following Proposition establishes the analytic and asymptotic properties of these functions.

Proposition 2.5. The coefficient functions $a_{ij}(z)$ satisfy the following properties: (i) The functions $a_{ij}(z)$ are analytic in $\mathbb{C} \setminus [0, 4]$. For $z < 0$, they are real-valued, positive, and monotonically increasing. (ii) For all $z \in \mathbb{R} \setminus [0, 4]$, the functions are related by the following identities:

$$a_{12}(z) = \left(1 - \frac{z}{2}\right) a_{11}(z) - \frac{1}{2}, \quad a_{22}(z) = \left(1 - \frac{z}{2}\right)^2 a_{11}(z) - \frac{2-z}{4}. \quad (2.3)$$

(iii) $\lim_{z \rightarrow -\infty} a_{ij}(z) = 0$. Furthermore, near the threshold $z = 0$, the following asymptotic expansions hold as $z \rightarrow 0^-$:

$$\begin{aligned} a_{11}(z) &= \frac{1}{2\sqrt{-z}} + O(1), \\ a_{12}(z) &= \frac{1}{2\sqrt{-z}} - \frac{1}{2} + O(\sqrt{-z}), \\ a_{22}(z) &= \frac{1}{2\sqrt{-z}} - \frac{1}{2} + O(\sqrt{-z}). \end{aligned}$$

Proof. (i) Analyticity is a direct consequence of the properties of the resolvent $(H_0 - zI)^{-1}$. For $z < 0$, the integrand in $a_{ij}(z)$ is positive since $\varepsilon(q) - z > 0$, ensuring positivity. Monotonicity follows from $\frac{d}{dz} a_{ij}(z) = \int_{\mathbb{T}} \frac{\alpha_i(q)\alpha_j(q)}{(\varepsilon(q) - z)^2} dq > 0$. (ii) Using $\cos q = 1 - \frac{\varepsilon(q)}{2}$, we observe that $\alpha_2(q) = \sqrt{2}\alpha_1(q)\cos q = \sqrt{2}\alpha_1(q)\left(1 - \frac{\varepsilon(q)}{2}\right)$. Substituting this into the definition of $a_{12}(z)$ and $a_{22}(z)$ and employing the identity $\frac{\varepsilon(q)}{\varepsilon(q) - z} = 1 + \frac{z}{\varepsilon(q) - z}$ leads to the algebraic relations in (2.3). (iii) The limit at $-\infty$ is obtained via the Lebesgue Dominated Convergence Theorem. The threshold asymptotics are derived from the explicit evaluation $a_{11}(z) = [2\sqrt{-z}\sqrt{4-z}]^{-1}$. Expanding this expression and the relations in (ii) around $z = 0$ yields the specified forms.

Lemma 2.6. The Fredholm determinant $\Delta_{\mu\lambda}(z)$ is well-defined and real-analytic for $z \in \mathbb{C} \setminus [0, 4]$. Moreover:

(1) The determinant satisfies $\lim_{z \rightarrow -\infty} \Delta_{\mu\lambda}(z) = 1$.

(2) Near the band edge $z = 0$, the following asymptotic expansion holds as $z \rightarrow 0^-$:

$$\Delta_{\mu\lambda}(z) = -\frac{\mu + 2\lambda + \mu\lambda}{2\sqrt{-z}} + \frac{4 + 2\mu + 2\lambda + \mu\lambda}{4} + O(\sqrt{-z}). \quad (2.4)$$

Proof. The analyticity and the limit at infinity follow from the definition $\Delta_{\mu\lambda}(z) = \det(I - B_{\mu\lambda}(z))$ and the properties of $a_{ij}(z)$ established in Proposition 2.5 To derive the expansion near $z = 0$, we substitute the asymptotics of $a_{ij}(z)$ into the expression:

$$\Delta_{\mu\lambda}(z) = (1 - \mu a_{11}(z))(1 - \lambda a_{22}(z)) - \mu\lambda a_{12}^2(z)$$

Collecting the leading terms of order $(-z)^{-1/2}$ and the constant terms $O(1)$ yields (2.4).

3. Main results and their proofs.

Lemma 3.1. The set of points in $\mathbb{R}^2$ satisfying the equation $\mu\lambda + \mu + 2\lambda = 0$ is the graph of the function

$$\mu(\lambda) = -\frac{2\lambda}{1 + \lambda}, \lambda \neq -1$$

This graph consists of two disjoint, smooth, and connected curves:

$$\ell_0 = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu = \mu(\lambda), \lambda > -1\}, \quad (3.1)$$

$$\ell_1 = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu = \mu(\lambda), \lambda < -1\}. \quad (3.1)$$

These curves partition the (λ, μ) -plane into three disjoint, connected regions:

$$\mathcal{C}_0 = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu > \mu(\lambda), \lambda > -1\},$$

$$\mathcal{C}_1 = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu < \mu(\lambda), \lambda > -1\} \cup \{(-1, \mu) : \mu \in \mathbb{R}\} \cup \{(\lambda, \mu) \in \mathbb{R}^2 : \mu > \mu(\lambda), \lambda < -1\},$$

$$\mathcal{C}_2 = \{(\lambda, \mu) \in \mathbb{R}^2 : \mu < \mu(\lambda), \lambda < -1\}.$$

Proof. The explicit form $\mu(\lambda)$ follows from basic algebraic rearrangement of $\mu\lambda + \mu + 2\lambda = 0$. The separation into ℓ_0 and ℓ_1 is due to the vertical asymptote at $\lambda = -1$. The connected components $\mathcal{C}_\alpha$ are then naturally defined by the strict inequalities and the line $\lambda = -1$, which joins the two regions "between" the hyperbolic branches. For further topological details, see [18].

We now analyze the number of bound states (isolated eigenvalues) of the operator $H_{\mu\lambda}$ corresponding to energies less than zero.

Theorem 3.2. Let $\mathcal{C} = \mathcal{C}_\alpha$, where $\alpha \in \{0, 1, 2\}$, denote the connected components defined by (3.2). Within any such component $\mathcal{C}$, the number of eigenvalues $n(H_{\mu\lambda})$ lying below the essential spectrum $\sigma_{\text{ess}}(H_{\mu\lambda})$ remains invariant. Moreover, this constant value is determined by:

$$n(H_{\mu\lambda}) = \alpha \text{ if } \mathcal{C} = \mathcal{C}_\alpha$$

Proof sketch. The result follows from the analyticity of the Fredholm determinant $\Delta_{\mu\lambda}(z)$ with respect to $z \in \mathbb{C} \setminus [0, 4]$ and its continuous dependence on the parameters (μ, λ) . For any fixed pair (μ, λ) not lying on the boundary curves ℓ_j , the function $z \mapsto \Delta_{\mu\lambda}(z)$ has no zeros at the threshold of the essential spectrum. According to the argument principle, the number of zeros in the interval $(-\infty, 0)$ is locally constant within the regions $\mathcal{C}_\alpha$. Since each region $\mathcal{C}_\alpha$ is connected, the number of eigenvalues $n_-(H_{\mu\lambda})$ remains invariant throughout each respective domain. The specific value of $\alpha \in \{0, 1, 2\}$ is determined by evaluating the determinant at representative points (for instance, $(\mu, \lambda) = (0, 0)$ for $\alpha = 0$) and considering the global constraint $n(H_{\mu\lambda}) \leq 2$ imposed by the rank-two nature of the perturbation. For a comprehensive derivation, we refer the reader to [16].

Theorem 3.3. Let $\alpha = 0, 1$ and $\mu, \lambda \in \mathbb{R}$. For any $(\mu, \lambda) \in \ell_\alpha, \alpha = 0, 1$, the number of negative eigenvalues of the operator $H_{\mu\lambda}$ remains constant throughout ℓ_α . Furthermore,

$$n(H_{\mu\lambda}) = \alpha \text{ for all } (H_{\mu\lambda}) \in \ell_\alpha.$$

Proof sketch of Theorem 3.3. The constancy of $n(H_{\mu\lambda})$ on each curve ℓ_α follows from the fact that the Fredholm determinant $\Delta_{\mu\lambda}(z)$ remains non-zero for all $z < 0$ as the parameters (μ, λ) vary along ℓ_α . Although $\Delta_{\mu\lambda}(0) = 0$ on these curves (signaling a virtual level or resonance at the threshold), this zero does not "enter" the discrete spectrum interval $(-\infty, 0)$ due to the specific asymptotic behavior of the matrix

elements $a_{ij}(z)$ near $z = 0$. By evaluating $n(H_{\mu\lambda})$ at a representative point on each curve ℓ_α , we obtain the constant values 0 and 1, respectively. $\square$

Theorems 3.2 and 3.3 collectively establish the necessary and sufficient conditions on the perturbation parameters (μ, λ) for the operator $H_{\mu\lambda}$ to possess exactly α discrete eigenvalues below the essential spectrum. This classification is summarized in the following corollary:

Corollary 3.4. For any $\alpha \in \{0, 1, 2\}$, the operator $H_{\mu\lambda}$ has exactly α eigenvalues (counting multiplicities) in the lower spectral gap $(-\infty, 0)$ if and only if

$$(\mu, \lambda) \in \mathcal{C}_\alpha \cup \ell_\alpha,$$

where the regions $\mathcal{C}_\alpha$ and the boundary curves ℓ_α are defined in (3.2) and (3.1), respectively, with the convention $\ell_2 = \emptyset$.

3.1. Critical Operators. To elucidate the mechanisms of eigenvalue birth and annihilation in the lower spectral gap $(-\infty, 0)$, we introduce the concept of a lower-critical parameter point.

Definition 3.5. A parameter point $(\mu_0, \lambda_0) \in \mathbb{R}^2$ is called lower-critical for $H_{\mu\lambda}$ if the number of discrete eigenvalues $n_-(H_{\mu\lambda})$ is non-constant in every neighborhood of (μ_0, λ_0) .

The following theorem characterizes the set of critical points through the coupling parameters.

Theorem 3.6. A parameter point $(\mu_0, \lambda_0) \in \mathbb{R}^2$ is lower-critical for $H_{\mu\lambda}$ if and only if

$$\mu_0 \lambda_0 + \mu_0 + 2\lambda_0 = 0.$$

Proof. By Corollary 3.4, the number of eigenvalues $n_-(H_{\mu\lambda})$ changes only when a zero of the Fredholm determinant $\Delta_{\mu\lambda}(z)$ emerges from or vanishes into the threshold of the essential spectrum at $z = 0$. From the asymptotic expansion (2.4), the leading singular term vanishes if and only if the coefficient satisfies:

$$\mu\lambda + \mu + 2\lambda = 0.$$

This condition ensures that $\Delta_{\mu\lambda}(z)$ can change its sign in a neighborhood of $z = 0$ for small perturbations of the parameters, which corresponds to the birth or annihilation of an eigenvalue. Thus, the set of lower-critical points coincide exactly with the boundary curves $\ell_0 \cup \ell_1$. $\square$

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FURYE INTEGRALLARI VA ULARNI TAQRIBIY HISOBLASH USULLARI

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Annotatsiya. Ushbu maqolada yuqori chastotali tebranuvchi integrallarni taqribiy hisoblashda Faylon tipidagi va asimptotik yoyish usullarining nazariy asoslari hamda ularning o‘zaro bog‘liqligi tahlil qilingan. Tadqiqotda klassik kvadratura formulalarining yuqori tebranishli funksiyalar uchun samarasizligi, shuningdek, asimptotik yoyish usulining chegaraviy nuqtalardagi xususiyatlari va Faylon usulida Ermit interpolatsiyasini qo‘llash orqali xatolik tartibini yaxshilash imkoniyatlari ko‘rsatilgan. Ikki usulning kombinatsiyalangan yondashuvi kichik va katta chastotalarda hisoblash aniqligini ta‘minlashi ko‘rsatilgan.

Kalit so‘zlar: Furiye almashtirishlari, asimptotik yoyish, Lagranj kardinal ko‘phadi, chegaraviy had, kvadratura formulalar, qoldiq had.

ИНТЕГРАЛЫ ФУРЬЕ И МЕТОДЫ ИХ ПРИБЛИЖЕННОГО ВЫЧИСЛЕНИЯ

Аннотация. В данной статье проанализированы теоретические основы методов типа Филон и асимптотического разложения для приближённого вычисления сильно осциллирующих интегралов, а также взаимосвязь между ними. В исследовании показана неэффективность классических квадратурных формул для функций с высокой частотой колебаний, а также особенности метода асимптотического разложения в граничных точках и возможности улучшения порядка погрешности путём применения эрмитовой интерполяции в методе Филон. Показано, что комбинированный подход двух методов обеспечивает вычислительную точность как при малых, так и при больших частотах.

Ключевые слова: преобразования Фурье, асимптотическое разложение, кардинальный многочлен Лагранжа, граничный член, квадратурные формулы, остаточный член.

FOURIER INTEGRALS AND METHODS FOR THEIR APPROXIMATE CALCULATION

Abstract. This article analyzes the theoretical foundations of Filon-type methods and asymptotic expansion methods for the approximate computation of highly oscillatory integrals, as well as the relationship between them. The study demonstrates the inefficiency of classical quadrature formulas for functions with high oscillation frequencies, the characteristics of the asymptotic expansion method at boundary points, and the possibilities of improving the error order by applying Hermite interpolation in the Filon method. It is shown that a combined approach of the two methods ensures computational accuracy at both small and large frequencies.

Keywords: Fourier transforms, asymptotic expansion, Lagrange cardinal polynomial, boundary term, quadrature formulas, remainder term.

Kirish. Bugungi zamonaviy raqamli texnologiyalar va sun‘iy intellekt davrida ma‘lumotlar hajmining keskin ortishi natijasida, ularni tahlil qilish uchun aniq, ishonchli va samarali matematik vositalarga bo‘lgan ehtiyoj ham sezilarli oshdi. Furiye qatorlari va integral almashtirishlari bu yo‘nalishda eng muhim usullardan biri bo‘lib kelmoqda. Furiye tahlili murakkab funksiyalarni sinus va kosinus to‘lqinlariga ajratish orqali ularning chastota spektrini aniqlash imkonini beradi. Bu usul asosan ikki shaklda qo‘llaniladi: davriy funksiyalar uchun Furiye qatorlari va nodavriy jarayonlar uchun Furiye integral almashtirishlari.

Hozirgi kunda Furiye almashtirishlari tibbiyot (MRI va KT), muhandislik, optika, telekommunikatsiya va sun‘iy intellekt kabi strategik sohalarning metodologik asosi hisoblanadi. Shuningdek, iqtisodiyot sohasida ushbu tahlil yordamida bozor sikllari va trendlarini prognoz qilish samaradorligi oshirishda ham samarali qo‘llaniladi.

Biroq, ko‘plab amaliy masalalarda Furiye integrallarini analitik ko‘rinishda hisoblashning imkoniyati bo‘lmaydi. Ayniqsa, integral ostidagi funksiyalar yuqori tebranish xususiyatiga ega bo‘lganda an‘anaviy usullar kutilgan aniqlikni bermaydi. Shu sababli, Furiye integrallarini taqribiy (sonli) hisoblashning samarali usullarini ishlab chiqish va ularni turli sohalarga tatbiq etish bugungi kunning dolzarb masalalaridan biri.

bo‘lib kelmoqda. Ushbu maqola Furiye integrallarining nazariy asoslarini tahlil qilish hamda ularning amaliy qo‘llanilish samaradorligini yoritishga bag‘ishlangan.

Asosiy qism. Furiye integrallari nazariyasining kelib chiqishini J. B. Furiyening "*Analytical Theory of Heat*" dan topishimiz mumkin. Faraz qilaylik, $2\pi\lambda$ davrli $f(x)$ funksiya Furiye qatorlarida ifodalangan bo‘lsin

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{nx}{\lambda} + b_n \sin \frac{nx}{\lambda} \right), \quad (1)$$

a_m, b_m koeffitsiyentlar mos ravishda $\cos\left(\frac{mx}{\lambda}\right)$ va $\sin\left(\frac{mx}{\lambda}\right)$ larga ko‘paytirilib $(-\pi\lambda, \pi\lambda)$ oraliqda integralni hisoblash orqali aniqlanadi

$$a_m = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t) \cos \frac{mt}{\lambda} dt, \quad b_m = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t) \sin \frac{mt}{\lambda} dt, \quad (2)$$

(1) va (2) formulalardan

$$f(x) = \frac{1}{2\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t) dt + \sum_{n=1}^{\infty} \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t) \cos \frac{n(x-t)}{\lambda} dt,$$

tenglikka ega bo‘lamiz.

Bu yerda $\pi/\lambda = u$, $1/\lambda = \delta u$ deb belgilab olsak, $\lambda \rightarrow \infty$ da yig‘indi ushbu integral ko‘rinishiga o‘tadi:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} du \int_{-\infty}^{\infty} f(t) \cos u(x-t) dt. \quad (3)$$

(3) ifodaga *Furiye integrali* deyiladi.

Bu formula Furiye qatorining davri cheksizlikka intilgandagi ($L \rightarrow \infty$) limiti natijasida hosil bo‘ladi. Uni yanada sodda va tushunarli ko‘rinishga keltirish uchun quyidagicha Furiye almashtirishlari shaklida yozish mumkin:

1. Furiye almashtirishi (Transform): $F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut} dt;$
2. Teskari Furiye almashtirishi: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) e^{-iux} du.$

Shuningdek, uni quyidagi (Furiye qatorlari) ko‘rinishda ham yozish mumkin

$$f(x) = \int_0^{\infty} \{a(u) \cos xu + b(u) \sin xu\} du. \quad (4)$$

Furiye integrallarini hisoblashning bir qancha samarali usullari bo‘lib, jumladan Faylon tipidagi usullar, Levin usuli, Statsionar faza usuli, asimptotik almashtirish usuli optimal kvadratura formulalari, Tezkor Furiye Almashtirishlari(FFT). Quyida biz ulardan Faylon va asimptotik yoyish usulining samaradorligi va o‘zaro bog‘liqligini ko‘ramiz.

Faylon usuli — kuchli tebranuvchi integrallarni (ayniqsa sinus/kosinusli) taqribiy hisoblash usuli sanaladi. U quyidagi shakldagi integrallar uchun ishlatiladi:

$$I = \int_a^b f(x) \cos(\omega x) dx \quad \text{yoki} \quad I = \int_a^b f(x) \sin(\omega x) dx.$$

Bunda $f(x)$ — silliq funksiya, $\omega \gg 1$ — katta chastotali parametr (ya’ni kuchli tebranish).

Klassik kvadratur formulalar (masalan, Simpson, Trapetsiya) tebranuvchi funksiya bilan ishlaganda **aniqlikni yo‘qotadi**. Faylon usulida esa:

- $f(x)$ funksiya ko‘phad bilan almashtiriladi (interpolyatsiya qilinadi);
- bu ko‘phad sinus yoki kosinusga ko‘paytiriladi;
- hosil bo‘lgan ifoda integrallash qoidasi asosida integrallanadi.

Biz quyidagi ko‘rinishdagi kuchli tebranuvchi integrallarni ko‘rib chiqamiz,

$$I[f] = \int_{-1}^1 f(x) e^{i\omega g(x)} dx. \quad (5)$$

Ma'lumki, odatiy Gauss kvadratur formulasi bu integralni taqribiy hisoblashda qo'llanilganda ω kattalashgani sari bu usulda xatolik $O(1)$ bo'ladi. ω katta bo'lganda $I[f]$ ni yaxshiroq yaqinlashtirish asimptotik yoyish orqali amalga oshiriladi. Faraz qilaylik, $g'(x) \neq 0, -1 < x \leq 1$, u holda qismlarga ajratib integrallashdan foydalanamiz:

$$I[f] = \frac{1}{i\omega} \left[\frac{f(1)}{g'(1)} e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)} e^{i\omega g(-1)} \right] - \frac{1}{i\omega} \int_{-1}^1 \frac{d}{dx} \left[\frac{f(x)}{g'(x)} \right] e^{i\omega g(x)} dx. \quad (6)$$

ω katta bo'lganda (2) tenglamadagi integral had chegaraviy hadlarga nisbatan tezroq yo'qoladi. Riemann-Lebesgue lemmasining yoyilmasiga ko'ra, chegaraviy hadlar integralni yaqinlashtira oladi. Shuningdek, jarayonni to'liq tasvirlash maqsadida integral qoldig'i asosida asimptotik yoyishni takroriy qo'llash mumkin. Asimptotik yoyishlar uchun odatdagidek, ketma-ketlikning farqlanishi sababli natijaning aniqligi cheklangan bo'ladi.

Yanada yaxshiroq yondashuv — kvadratur formula tugunlari $c_1, \dots, c_\nu$ ni tanlab, f funksiyani shu nuqtalarda interpolatsiya qilib, ko'phad $\tilde{f}$ bilan almashtirishdir:

$$Q_1^F[f] = \int_{-1}^1 \tilde{f}(x) e^{i\omega g(x)} dx = \sum_{j=1}^{\nu} b_j(\omega) f(c_j),$$

bu yerda $b_j(\omega) = \int_{-1}^1 l_j(x) e^{i\omega g(x)} dx$ had $l_j(x)$ j-Lagranj kardinal ko'phadi.

Ushbu yondashuvning bir varianti — agar $g(x) = x$ bo'lsa, Faylon usulida bo'laklab kvadratik interpolatsiyadan foydalanish L.N.G. Faylon [1] ga borib taqaladi. Bu turdagi sxemalar Faylon tipidagi usullar deb ataladi. $b_j(\omega)$ ni qurish uchun $\int_{-1}^1 x^m e^{i\omega g(x)} dx$ momentlar kerak bo'ladi. Bu momentlar o'zlari ham tebranuvchi integrallar bo'lib, ma'lumki, ularni Furrye holatidagi kabi analitik usul bilan hisoblash mumkin. Aks holda, Faylon usuli uchun momentlarni hisoblashda sonli eng tez tushish usuli (steepest descent method) dan foydalanish mumkin — bu usul amaliy qo'llanmalarda yaxshi natija beradi [2-3]. A.Iserles [4] shuni isbotlaganki, agar intervalning uchlari kvadratur formula tugunlari sifatida olingan bo'lsa va $g'(x) \neq 0, -1 \leq x \leq 1$, u holda bu yondashuv quyidagi xatolikka olib keladi:

$$Q_1^F[f] - I[f] = O(\omega^{-2}), \quad \omega \rightarrow \infty.$$

Bu yondashuvning asimptotik yoyilmaga nisbatan ustunligi shundaki, ω kattaligidan qat'i nazar ushbu usul muammolarning bir sinfi uchun aniq yechim beradi, ω katta bo'lgandagi qiymatlar uchun A.Iserles va S.P.Nørsett [6] isbotlashicha, agar Ermit interpolatsiyasi yordamida $f(x)$ ni oxirgi nuqtalarda p -tartibli hosilalari bilan interpolatsiya qilinsa, xatolikning asimptotik xatti-harakati quyidagicha ifodalanadi:

$$Q_p^F[f] - I[F] = O(\omega^{-p-1}), \quad \omega \rightarrow \infty,$$

g statsionar nuqtalar ξ nuqtalar bilan $g'(\xi) = 0$ bo'lgan holatda ushbu nazariyani kengaytirish mumkin. Yaxshi asimptotik xossalarga erishish uchun bajarilishi kerak bo'lgan narsa — bu statsionar nuqtalarni kvadratur tugunlari ichiga kiritishdir [5].

Asimptotik yoyilmani qoldiq had bilan birga ko'rib chiqilganda, muammo aslida chegaraviy hadlar va qoldiq hadga aylanganini ko'rish mumkin. Bu qoldiq had asl holatdagidek ifodalanadigan integral shaklidir. Shu kuzatuv asosida tabiiy savol tug'iladi: agar qolgan had maxsus texnika — masalan, Faylon tipidagi kvadratur, sonli tez tushish usuli yoki Levin usuli orqali ko'rib chiqilsa, aniqlik yaxshilanadi, bu quyidagi metodga olib keladi:

$$Q^{FA}[f] = \frac{1}{i\omega} \left[\frac{f(1)}{g'(1)} e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)} e^{i\omega g(-1)} \right] - \frac{1}{i\omega} Q_p^F \left[\frac{d}{dx} [f/g'] \right].$$

Bu usulda Faylon/asimptotik yoyish usullari birlashtirilgan holda taqdim qilingan.

Shuni ta'kidlash kerakki, $\omega \neq 0$ holatida bu usul izchil natijalar beradi, chunki qoldiq had uchun

yanada aniqroq kvadratur formulalar usulidan foydalanish orqali aniqlikni oshirish mumkin. Bu esa asimptotik yoyishda mavjud bo'lmagan muhim afzallikdir.

Bundan tashqari, ifodada mavjud bo'lgan $\frac{1}{\omega}$ omili tufayli, bu usulning asimptotik xatolik xatti-harakati to'g'ridan-to'g'ri qo'llaniladigan klassik Faylon tipidagi usulga nisbatan ancha yaxshilanadi. Bu shuni anglatadiki, yuqori asimptotik aniqlikka erishish uchun kamroq miqdorda momentlarni hisoblash kifoya qiladi.

Asimptotik yoyish usuli: Biz tavsifni birlashtirilgan usulning asosiy tarkibiy qismlarining umumiy ko'rinishini bayon etishdan boshlaymiz. Bu yuqori chastotali (tebranuvchi) integrallarning asimptotik yoyilmasi va Faylon tipidagi metodlarning kombinatsiyasidan iborat bo'lishi mumkin.

Quyida yuqori chastotali kvadratur formulalar usulini $Q_p[f] \approx I[f]$ ko'rinishida belgilaymiz, bu yerda p asimptotik tartibni bildiradi. Bu usul silliq f funksiyalar uchun qo'llaniladi va xatolikning kamayishini ta'minlaydi

$$Q_p[f] - I[f] = O(\omega^{-p-1}), \quad \omega \rightarrow \infty.$$

Shuni ta'kidlash kerakki, ayrim adabiyot manbalarida bu tartib $p+1$ sifatida ko'rsatiladi, bu absolyut xatolikning kamayish tezligiga mos keladi. Biroq, bizning holatimizda — agar statsionar (barqaror) nuqtalar mavjud bo'lmasa — xatolikning kamayishi $I[f] = 1/\omega$ nisbiy ko'rinishda ifodalanadi. Agar statsionar nuqtalar mavjud bo'lsa, vaziyat bir oz farq qiladi, shu sababli soddalik nuqtai nazaridan asimptotik tartib tushunchasiga murojaat qilmaymiz.

Ayaylik biz qarayotgan $g'(x) \neq 0$, $-1 \leq x \leq 1$ oraliqda statsionar nuqtalar yo'q bo'lsin. (1) yuqori tebranuvchi integralni asimptotik yoyish har bir qism bo'yicha integratsiyani ketma-ket qo'llash orqali olinadi. Ushbu yondashuv bizga quyida ko'rsatilgan qisman kengaytma orqali ifodani to'liq kengaytirish imkonini beradi.

$$I[f] = - \sum_{m=1}^s \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right] + \frac{1}{(-i\omega)^s} \int_{-1}^1 \sigma_s[f](x) e^{i\omega g(x)} dx, \quad (7)$$

Ta'rif 1: Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[-1, 1]$ oraliqda yetarli darajada silliq funksiyalar bo'lsin va $g'(x) \neq 0$ bo'lsin. U holda $\sigma_m[f]$ operatori quyidagicha rekursiv aniqlanadi

$$\begin{aligned} \sigma_0[f](x) &= f(x), \\ \sigma_{m+1}[f](x) &= \frac{d}{dx} \frac{\sigma_m[f](x)}{g'(x)}, \quad k = 0, 1, \dots \end{aligned} \quad (8)$$

Induksiya usuli yordamida yuqoridagi yoyishning to'g'riligini osongina tekshirish mumkin.

$$I[f] = - \sum_{m=1}^{\infty} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \quad (9)$$

Qatorni s haddan keyin qisqartirish asimptotik yondashuvni hosil qiladi

$$Q_s^A[f] = - \sum_{m=1}^s \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \quad (10)$$

Bu s -tartibli asimptotik yoyish usuli hisoblanadi. Buni $O(\omega^{-1})$ tebranish integralining $(-i\omega)^{-s}$ ga ko'paytiriladigan qoldiq had orqali yozish mumkin. Shuningdek, asimptotik tartib tushunchasi har doim ham katta ahamiyatga ega emas va ayrim hollarda foydali bo'lmashligi mumkin. Umuman olganda, asimptotik yoyish ko'pincha yaqinlashmaydigan xususiyatga ega (divergent) bo'ladi [7]. Kichikroq ω qiymatlari uchun bu jiddiyroq muammoga aylanishi mumkin. Shuning uchun, asimptotik yoyish usuli yetarlicha katta ω uchun foydalibroq bo'ladi. Bundan tashqari, farq faqat belgilangan aniqlikka erishish mumkinligini anglatadi – terminlar qo'shilishi har doim aniqlikni oshirmaydi. Amaliy ilovalarda, ayniqsa ma'lum aniqlik talab qilinadigan hollarda bu muammo bo'lishi mumkin.

Faylon tipidagi usullar kichikroq xatolik beradi va yaxshi boshqariladi, lekin hisoblash bahosi yuqori bo'lishi mumkin. Biz bu yerda momentlar aniqlikni maqsad qilamiz:

Masala 1: Ushbu integralni Faylon usuli yordamida yeching:

$$\mu_k(\omega) = \int_{-1}^1 x_k e^{i\omega g(x)} dx.$$

Yechim: $-1 = c_1 < c_2 < \dots < c_v = 1$ qilib tugunlar to'plami tanlanadi. Har bir tugunga tegishli $m_1, \dots, m_v \geq 1$ butun sonli ko'paytmalar orqali $n = \sum_{j=1}^v m_j - 1$ qilib tanlanadi va $\tilde{f}$ ni $\{c_j\}_{j=1}^v$ nuqtalarda tegishli nuqtalarda interpolatsiya qilish natijasida olingan n darajadagi yagona Ermit interpolatsiya ko'phadi tuziladi:

$$\tilde{f} = \sum_{l=1}^v \sum_{j=0}^{m_l-1} \alpha_{l,j}(x) f^{(j)}(c_l).$$

Faylon tipidagi usul quyidagicha aniqlanadi:

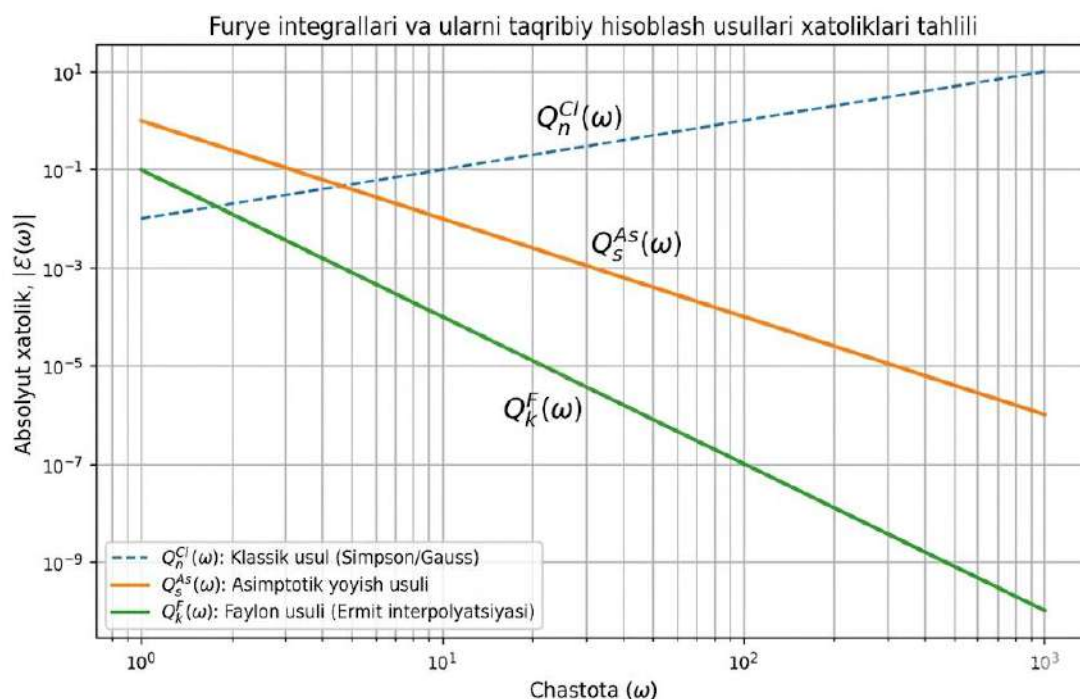
$$Q_s^F[f] = \int_{-1}^1 \tilde{f}(x) e^{i\omega g(x)} dx = \sum_{l=1}^v \sum_{j=0}^{m_l-1} \beta_{l,j}(\omega) f^{(j)}(c_l), \quad (11)$$

bu yerda $\beta_{l,j}(\omega) = \int_{-1}^1 \alpha_{l,j}(x) e^{i\omega g(x)} dx$ momentlarning chiziqli kombinatsiyasi orqali olinadi. s ga

kelsak, ushbu usulning asimptotik tartibida biz A.Iserles va S.P .Norsett [6] tufayli ushbu teoremani keltiramiz:

Teorema 1. Faraz qilaylik, $m_1, m_v \geq s$ bo'lsin, u holda har bir silliq f hamda silliq g $g'(x) \neq 0$, $-1 \leq x \leq 1$ uchun quyidagi munosabat o'rinli bo'ladi

$$Q_s^F[f] - I[f] = O(\omega^{-s-1}), \quad \omega \rightarrow \infty.$$



1-rasm

Ushbu teoremadan shunday xulosa qilish mumkinki, faqat oraliqning chetki nuqtalarida hosilalarning mavjudligi haqidagi ma'lumot yordamida asimptotik tartibni yaxshilash mumkin. Hosilalar haqida ma'lumot, shuningdek, chetki nuqtalar yaqinidagi interpolatsiya tugunlarini klasterlash orqali bilvosita ta'minlanishi mumkin. Agar tugunlar chetki nuqtalarga $1/\omega$ ga yaqinlashsa, yuqori asimptotik tartibga erishish mumkin [8].

Yuqorida tahlil qilinganidek, 1-rasmdan ko'rinib turibdiki, an'anaviy usullardan farqli o'laroq, Faylon va asimptotik yoyish usullari tebranishlar chastotasi ortgan sari yuqori aniqlikni ta'minlaydi.

Xulosa. Tadqiqotda klassik kvadratura formulalari (Simpson, Gauss) chastota ω ortishi bilan $O(1)$ tartibli xatolik berib, samarasiz bo'lib qolishi sababli, yuqori chastotali tebranuvchi integrallarni hisoblashning Faylon va asimptotik yoyish usullari nazariy jihatdan tahlil qilindi. Asimptotik yoyish usuli katta ω da $O(\omega^{-s})$ aniqlik ta'minlasa-da, divergent xususiyati va kichik ω da ishonchsizligi uning asosiy kamchiligi hisoblanadi. Faylon usulida Ermit interpoliyasini qo'llash xatolikni $O(\omega^{-s-1})$ ga tushirib, bir tartib yuqori aniqlik beradi.

Ikki usulning kombinatsiyasi, ya'ni Q^{FA} yondashuvi ham kichik, ham katta chastotalarda barqaror aniqlikni ta'minlaydi. Olingan natijalar Fyurje almashtirishlari, to'liqin tarqalishi va tibbiy tasvirlash kabi sohalarda yuqori aniqlikdagi algoritmlar yaratish uchun asos bo'lib xizmat qiladi.

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TRANSCENDENT TENGLAMA VA TENGSIZLIKLAR

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Annotatsiya. Ushbu maqolada transcendental tenglamalar va tengsizliklar tushunchalari, ularning turlari va yechish usullari o'rganiladi. Transcendental tenglamalar – bu algebraik emas, ya'ni o'zida trigonometrik, ko'rsatkichli, logarifmik yoki aralash funksiyalarni o'z ichiga olgan tenglamalardir. Ushbu tenglamalarni yechish an'anaviy algebraik usullarga qaraganda murakkab bo'lib, ko'pincha grafik, analitik va sonli usullar yordamida amalga oshiriladi. Shuningdek, transcendental tengsizliklar ham xuddi shunday murakkablikka ega bo'lib, ularni yechish uchun funksiya xatti-harakatlarini tahlil qilish, hosila olish va intervallar bo'yicha tekshirish kabi metodlar qo'llanadi. Ishda turli misollar orqali tenglamalar va tengsizliklarning yechish algoritmlari ko'rsatilib, amaliy yechim strategiyalari taqdim etiladi.

Kalit so'zlar: transcendent tenglama, transcendent tengsizlik, ko'rsatkichli funksiya, logarifmik funksiya, trigonometrik funksiya, grafik usul.

ТРАНЦЕНДЕНТНЫЕ УРАВНЕНИЯ И НЕРАВЕНСТВА

Аннотация. В данной статье рассматриваются понятия трансцендентных уравнений и неравенств, их типы и методы решения. Трансцендентные уравнения – это неалгебраические уравнения, включающие тригонометрические, показательную, логарифмическую или смешанную функции. Решение таких уравнений сложнее традиционных алгебраических методов и чаще всего выполняется с использованием графических, аналитических и численных методов. Трансцендентные неравенства обладают аналогичной сложностью, и для их решения применяются методы анализа поведения функции, вычисления производной и проверки на интервалах. В статье приведены различные примеры алгоритмов решения уравнений и неравенств, а также даны практические стратегии решения.

Ключевые слова: трансцендентное уравнение, трансцендентное неравенство, показательная функция, логарифмическая функция, тригонометрическая функция, графический метод.

TRANSCENDENT EQUATIONS AND INEQUALITIES

Abstract. This article explores the concepts of transcendental equations and inequalities, their types, and methods of solution. Transcendental equations are non-algebraic, containing trigonometric, exponential, logarithmic, or mixed functions. Solving these equations is more complex than traditional algebraic methods and is often carried out using graphical, analytical, and numerical approaches. Transcendental inequalities have similar complexity, and solving them involves analyzing the function's behavior, computing derivatives, and checking intervals. The article provides various examples of solution algorithms for equations and inequalities and presents practical solving strategies.

Keywords: transcendental equation, transcendental inequality, exponential function, logarithmic function, trigonometric function, graphical method.

$f(x) = g(x)$ tenglamada $f(x)$ yoki $g(x)$ funksiyalardan hech bo'lmaganda bittasi algebraik bo'lmasa bu tenglama transcendent tenglama deyiladi. Umuman olganda, trigonometrik, logarifmik va ko'rsatkichli tenglamalar transcendent tenglamalar deyiladi. Ba'zi hollarda ular algebraik tenglamalarni yechishga keltiriladi. Tenglamaning har ikki qismida turli xil funksiyalar turgan hollarni ko'ramiz. Masalan, $f(x)$ funksiya trigonometrik, $g(x)$ funksiya logarifmik funksiyalar bo'lsin. Transcendent tenglamalarni teng kuchli algebraik shakl almashtirishlar bajarib yechib bo'lmaydi. Bunday tenglamalarni yechishda matematik analiz elementlaridan foydalaniladi. Aniq masalalar keltiramiz.

1-misol. Ushbu $\log_2(x^2 - 4x + 8) = 1 + \sin(x - 2)$ tenglamani yeching.

Yechish. $0 \leq 1 + \sin(x - 2) \leq 2$, shuning uchun

$$0 \leq \log_2(x^2 - 4x + 8) \leq 2, \quad 1 \leq x^2 - 4x + 8 \leq 4.$$

$$\begin{cases} x^2 - 4x + 7 \geq 0 \\ (x - 2)^2 \leq 0 \end{cases}$$

Bundan $x = 2$. Tekshirish $x = 2$ berilgan tenglama ildizi bo'lishini ko'rsatadi.

Javob: 2

2-misol. Ushbu $\sqrt{\log_2 \sin x} + \cos x = x - \frac{\pi}{2}$ tenglamani yeching.

Yechish. $\log_2 \sin x \geq 0 \Leftrightarrow \log_2 \sin x \geq \log_2 1; \sin x \geq 1, -1 \leq \sin x \leq 1$ bo'lganligi

uchun $\sin x = 1$, u holda $\cos x = 0$ va $x = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$. Demak, $x - \frac{\pi}{2} = 0$, shuning uchun

$x = \frac{\pi}{2}$ berilgan tenglamaning yagona ildizidir.

Javob: $\frac{\pi}{2}$

3-misol. Ushbu $\log_{\sin x} \left(\sin x - \frac{1}{4} \cos x \right) = 3$ tenglamani yeching.

Yechish. Tenglamaning aniqlanish sohasi

$$\begin{cases} 0 < \sin x < 1, \\ \sin x - \frac{1}{4} \cos x > 0. \end{cases}$$

dan iborat. Bu sohada berilgan tengsizlikni ushbu

$$\log_{\sin x} \left(\sin x - \frac{1}{4} \cos x \right) = \log_{\sin x} \sin^3 x$$

ko'rinishda yozib olib, so'ngra potentsirlab

$$\sin x - \frac{1}{4} \cos x = \sin^3 x,$$

tenglamaga ega bo'lamiz va yechamiz:

$$\sin x (1 - \sin^2 x) - \frac{1}{4} \cos x = 0,$$

$$\cos x \left(\sin x \cos x - \frac{1}{4} \right) = 0,$$

$$\cos x \left(\sin 2x - \frac{1}{2} \right) = 0.$$

Berilgan tenglama aniqlanish sohasida $\cos x \neq 0$ bo'lganligi uchun

$$\sin 2x - \frac{1}{2} = 0$$

bundan

$$x_1 = \frac{\pi}{12} + \pi k, \quad x_2 = \frac{5\pi}{12} + \pi k, \quad k \in \mathbb{Z}$$

ni topamiz. Tenglamaning aniqlanish sohasiga ko'ra bu ildizlarning faqatgina

$$x_1 = \frac{\pi}{12} + 2\pi k, \quad x_2 = \frac{5\pi}{12} + 2\pi k, \quad k \in \mathbb{Z}$$

ko'rinishi berilgan tenglamani qanoatlantiradi.

$$\text{Javob: } x_1 = \frac{\pi}{12} + 2\pi k, \quad x_2 = \frac{5\pi}{12} + 2\pi k, \quad k \in \mathbb{Z}$$

4-misol. Ushbu $\log_2 \left(\cos^2 xy + \frac{1}{\cos^2 xy} \right) = \frac{1}{y^2 - 2y + 2}$ tenglamani yeching.

Yechish. $0 < a \neq 1$ da $a + \frac{1}{a} > 2$ va $a = 1$ da $a + \frac{1}{a} = 2$ bo'lganligi uchun

$$\cos^2 xy + \frac{1}{\cos^2 xy} \geq 2$$

Shuning uchun

$$\log_2 \left(\cos^2 xy + \frac{1}{\cos^2 xy} \right) \geq 1$$

Tenglik belgisi $\cos^2 xy = 1$ da o'rinli. 2-tomondan

$$\frac{1}{y^2 - 2y + 2} = \frac{1}{(y-1)^2 + 1} \leq 1$$

Bunda tenglik faqat $y = 1$ da o'rinli. Demak, berilgan tenglama yechimga ega bo'lishi uchun x va y lar quyidagi tenglamalar sistemasini qanoatlantirishi kerak.

$$\begin{cases} \cos^2 xy = 1 \\ y = 1 \end{cases}$$

Bu sistemani yechib $x = \pi k, k \in \mathbb{Z}; y = 1$ ni hosil qilamiz.

$$\text{Javob: } x = \pi k, \quad k \in \mathbb{Z}; \quad y = 1$$

5-misol. Ushbu $(x-3)\log_4 x + (x^2 - 4x + 2)\log_4 x + 1 - x = 0$ tenglamaning butun ildizlarini toping.

Yechish. $\log_4 x = t$ belgilash kiritamiz va berilgan tenglamani t ga nisbatan kvadrat tenglama sifatida yechamiz:

$$(x-3)t + (x^2 - 4x + 2)t + 1 - x = 0$$

Uning ildizlarini topamiz. $x = 3$ tenglama ildizi bo'lganligi uchun kvadrat tenglama

$$t^2 - \left(1 - x + \frac{1}{x-3} \right) t + \frac{1-x}{x-3} = 0$$

tenglamaga ekvivalent. Viyet teoremasiga ko'ra, $t_1 = 1 - x; t_2 = \frac{1}{x-3}$. Oxirgi tenglamaning chap qismini ko'paytuvchilarga ajratamiz:

$$(t + x - 1) \left(t - \frac{1}{x-3} \right) = 0$$

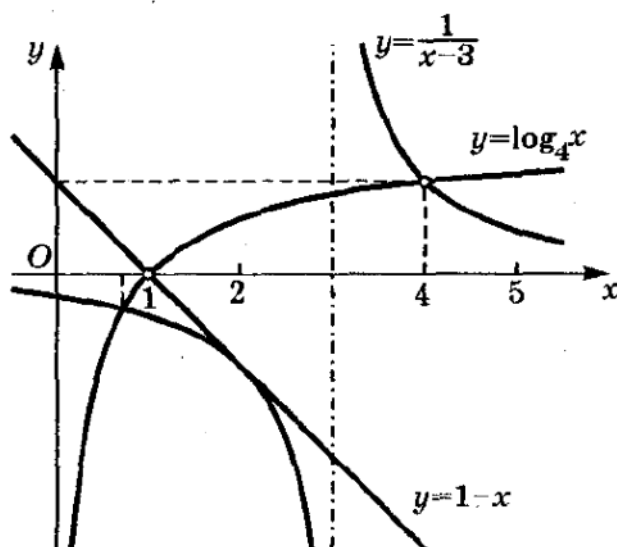
Belgilashga qaytib

$$(\log_4 x + x - 1) \left(\log_4 x - \frac{1}{x-3} \right) = 0$$

tenglamani hosil qilamiz. Bu tenglama quyidagi tenglamalar birlashmasiga teng kuchli

$$\log_4 x = 1 - x \text{ va } \log_4 x = \frac{1}{x-3}$$

Birinchi tenglama $x=1$, ikkinchi tenglama $x=4$ butun ildizga ega (1-chizma)



1-chizma

Javob: $x_1 = 1, x_2 = 2$

6-misol. Ushbu

$$(\log_{tgx} 3)^2 \leq \log_{tgx} (3tg^2 x)$$

tengsizlikni yeching.

Yechish. Tengsizlikning aniqlanish sohasi $0 < tgx \neq 1$ dan iborat bo'lganligi uchun berilgan tengsizlik

$$(\log_{tgx} 3)^2 \leq \log_{tgx} 3 + 2$$

dan iborat. Bundan $-1 \leq \log_{tgx} 3 \leq 2$. Oxirgi qo'sh tengsizlik quyidagi 2 ta sistemalar birlashmasiga teng kuchli.

$$\begin{cases} tgx > 1, \\ \frac{1}{tgx} \leq 3 \leq tg^2 x; \end{cases} \quad (I)$$

$$\begin{cases} 0 < tgx < 1 \\ \frac{1}{tgx} \geq 3 \geq tg^2 x. \end{cases} \quad (II)$$

(I) sistema quyidagi ko'rinishga ega bo'ladi:

$$\begin{cases} tgx > 1, \\ tgx \geq \frac{1}{3}, \\ tgx \geq \sqrt{3}. \end{cases}$$

Bu $tgx \geq \sqrt{3}$ tengsizlikka teng kuchli. Bundan $\frac{\pi}{3} + \pi k \leq x < \frac{\pi}{2} + \pi k; k \in \mathbb{Z}$.

(II) sistema quyidagi ko'rinishga ega bo'ladi.

$$\begin{cases} 0 < \operatorname{tg} x < 1, \\ \operatorname{tg} x \leq \frac{1}{3}, \\ \operatorname{tg} x \leq \sqrt{3}. \end{cases}$$

U $0 < \operatorname{tg} x \leq \frac{1}{3}$ sistemaga ekvivalent. Bundan $k\pi < x \leq \operatorname{arctg} \frac{1}{3} + k\pi, \quad k \in \mathbb{Z}$

$$\text{Javob: } \frac{\pi}{3} + \pi k \leq x < \frac{\pi}{2} + \pi k, \pi k < x \leq \operatorname{arctg} \frac{1}{3} + \pi k; \quad k \in \mathbb{Z}$$

7-misol. Ushbu $\cos^2(x+1) \lg(9-2x-x^2) \geq 1$ tengsizlikni yeching.

Yechish. Tengsizlikning chap qismi 2 ta ko'paytuvchidan iborat. Ixtiyoriy x uchun

$$0 \leq \cos^2(x+1) \leq 1$$

tengsizlik o'rinli. Bu ko'paytma

$$\lg(9-2x-x^2) \geq 1$$

bo'lganda birga teng yoki birdan katta bo'ladi.

$$\begin{cases} (9-2x-x^2) \geq 10, \\ (x+1)^2 \leq 0. \end{cases}$$

Bundan $x = -1$. Tekshirish natijasida $x = -1$ berilgan tengsizlikning yechimi bo'ladi.

Javob: $x = -1$

8-misol. Ushbu $\left(1 - \frac{x}{2}\right) \log_{13-3 \cdot 2^x} 4 \leq 1$ tengsizlikni yeching.

Yechish. Berilgan tengsizlikni yechish quyidagi tengsizlikni yechishga keltiriladi:

$$f(x) = \left(1 - \frac{x}{2}\right) \log_{13-3 \cdot 2^x} 4 - 1 = \frac{(2-x) - \log_2(13-3 \cdot 2^x)}{\log_2(13-3 \cdot 2^x)} \leq 0$$

$f(x)$ funksiyani aniqlanish sohasini topamiz:

$$0 < 13 - 3 \cdot 2^x \neq 1$$

$$2^x \neq 4; \quad 2^x < \frac{13}{3}, \quad x \neq 2; \quad x < \log_2 \frac{13}{3}$$

Endi $f(x)$ funksiya suratining nollarini topamiz:

$$(2-x) - \log_2(13-3 \cdot 2^x) = 0$$

$$\log_2 2^{2-x} = \log_2(13-3 \cdot 2^x)$$

$$\left(2^x - \frac{1}{3}\right)(2^x - 4) = 0 \Leftrightarrow \begin{cases} x = \log_2 \frac{1}{3} \\ x = 2 \end{cases}$$

Topilgan qiymatlar funksiya aniqlanish sohasini quyidagi intervallarga ajratadi:

$$\left(-\infty; \log_2 \frac{1}{3}\right), \left(\log_2 \frac{1}{3}; 2\right), \left(2; \log_2 \frac{13}{3}\right)$$

Bu intervallarning har birida $f(x)$ funksiyaning ishorasini aniqlaymiz. Buning uchun $f(x)$ funksiyaning quyidagi ko'rinishda yozib olamiz:

$$f(x) = \frac{\varphi_1(x)}{\varphi_2(x)}$$

Bunda

$$\varphi_1(x) = (2-x) - \log_2(13-3 \cdot 2^x)$$

$$\varphi_2(x) = \log_2(13-3 \cdot 2^x)$$

Quyidagi jadvalni to'ldiramiz

№	1	2	3
Interval	$\left(-\infty; \log_2 \frac{1}{3}\right)$	$\left(\log_2 \frac{1}{3}; 2\right)$	$\left(2; \log_2 \frac{13}{3}\right)$
$\varphi_1(x)$ ishorasi	+	-	+
$\varphi_2(x)$ ishorasi	+	+	-
$f(x)$ ishorasi	+	-	-

Demak, berilgan tengsizlik yechimi jadvaldagi ikkinchi va uchinchi interval va funksiyaning noli $x = \log_2 \frac{1}{3}$ dan iborat.

$$\text{Javob: } \left[\log_2 \frac{1}{3}; 2\right) \cup \left(2; \log_2 \frac{13}{3}\right)$$

9-misol. Ushbu $\log_{84-2x-x^2} \cos x \leq \log_{x+19} \cos x$ tengsizlikni yeching.

Yechish. Berilgan tengsizlikni

$$f(x) = \log_{84-2x-x^2} \cos x - \log_{x+19} \cos x \leq 0$$

tengsizlikni yechishga keltiramiz. Funksiyaning aniqlanish sohasini topamiz.

$$\begin{cases} 0 < 84 - 2x - x^2 \neq 1, \\ 0 < x + 19 \neq 1, \\ \cos x > 0 \end{cases} \Leftrightarrow \begin{cases} -7 < x < x_1, \\ x_1 < x < -\frac{3\pi}{2}, \\ -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \frac{3\pi}{2} < x < x_2, \\ x_2 < x < 6. \end{cases}$$

$$\text{Bunda } x_1 = -\frac{1+\sqrt{167}}{2} \text{ va } x_2 = \frac{-1+\sqrt{167}}{2}. \text{ Endi funksiyaning nollarini topamiz.}$$

Funksiyaning aniqlanish sohasida bu tenglama quyidagi tenglamalar birlashmasiga teng kuchli

$$\begin{cases} \cos x = 1, \\ 84 - 2x - 2x^2 = x + 19 \end{cases} \Leftrightarrow \begin{cases} x = -2\pi, \\ x = 0, \\ x = -\frac{13}{2}, \\ x = 5 \end{cases}$$

$f(x)$ funksiyaning 9 ta ko'rsatilgan intervallarda ishoralarini aniqlash uchun quyidagi jadvalni tuzamiz. Qaralayotgan intervallarda $0 < \cos x < 1$ bo'lganligi uchun funktsiyani quyidagi ko'rinishda yozish mumkin.

$$f(x) = \frac{\log_{\cos x}(x+19) - \log_{\cos x}(84 - 2x - 2x^2)}{\log_{\cos x}(84 - 2x - 2x^2) \log_{\cos x}(x+19)}$$

yoki

$$f(x) = \frac{\varphi_1(x)}{\varphi_2(x)\varphi_3(x)}$$

Bunda

$$\begin{cases} \varphi_1(x) = \log_{\cos x}(x+19) - \log_{\cos x}(84 - 2x - 2x^2) \\ \varphi_2(x) = \log_{\cos x}(84 - 2x - 2x^2) \\ \varphi_3(x) = \log_{\cos x}(x+19) \end{cases}$$

Bu funksiyalarning ishoralarini har bir qaralayotgan intervallarda aniqlaymiz.

№	1	2	3	4	5	6	7	8	9
Intervallar	$(-7; x_1)$	$(x_1; -\frac{13}{2})$	$(-\frac{13}{2}; -2\pi)$	$(-2\pi; -\frac{3\pi}{2})$	$(-\frac{\pi}{2}; 0)$	$(0; \frac{\pi}{2})$	$(\frac{3\pi}{2}; 5)$	$(5; x_2)$	$(x_2; 6)$
$\varphi_1(x)$ ishorasi	-	-	+	+	+	+	+	-	-
$\varphi_2(x)$ ishorasi	+	-	-	-	-	-	-	-	+
$\varphi_3(x)$ Ishorasi	-	-	-	-	-	-	-	-	-
$f(x)$ ishorasi	+	-	+	+	+	+	+	-	+

Jadvaldan ko'rinib turibdiki, $f(x)$ funksiya manfiy qiymatlarni ikkinchi va sakkizinchi intervallarda qabul qiladi. Funktsiya nollarini qo'shib tengsizlik yechimini hosil qilamiz.

$$\text{Javob: } \left(-\frac{1+\sqrt{167}}{2}; -\frac{13}{2}\right) \cup \left[5; \frac{-1+\sqrt{167}}{2}\right) \cup \{-2\pi\} \cup \{0\}.$$

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KASR TARTIBLI DIFFUZIYA TENGLAMASI UCHUN KOEFFITSIYENTNI ANIQLASHNING TESKARI MASALASI

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Annotatsiya. Mazkur ishda cheksiz sohada qaraladigan kasr tartibli diffuziya tenglamasi uchun vaqtga bog‘liq ko‘effitsiyentni aniqlash teskari masalasi o‘rganiladi. Tenglamada Riman–Liuvill kasr tartibli

hosilasi ishtirok etadi. Dastlab $q(t) \equiv 0$ hol uchun ma‘lum aniq yechimlardan foydalanib, ko‘effitsiyent mavjud bo‘lgan hol uchun Koshi masalasining mavjudligi va yagonaligi tadqiq qilinadi. Shuningdek, qo‘shimcha

$u(0,t)=g(t)$ shart asosida noma‘lum ko‘effitsiyentni tiklash masalasi qo‘yiladi. Teskari masalaning yechimining mavjudligi va yagonaligi Banaxning qo‘zg‘almas nuqta prinsipi yordamida asoslanadi. Olingan natijalar kasr tartibli tenglamalar nazariyasini rivojlantirishda muhim ahamiyatga ega.

Kalit so‘zlar: kasr tartibli diffuziya tenglamasi, Riman–Liuvill hosilasi, teskari masala, Koshi masalasi, Banax qo‘zg‘almas nuqta prinsipi, mavjudlik, yagonalik, ko‘effitsiyentni tiklash.

ОБРАТНАЯ ЗАДАЧА ОПРЕДЕЛЕНИЯ КОЭФФИЦИЕНТА В ДРОБНО-ДИФФУЗИОННОМ УРАВНЕНИИ

Аннотация. В данной работе исследуется обратная задача определения зависящего от времени коэффициента для дробно-диффузионного уравнения в бесконечной области. Уравнение содержит дробную производную по времени в смысле Римана–Лиувилля. Сначала, используя точные

решения для случая $q(t) \equiv 0$, изучаются вопросы существования и единственности решения задачи Коши при наличии коэффициента. Кроме того, формулируется обратная задача восстановления

неизвестного коэффициента с использованием дополнительного условия $u(0,t)=g(t)$. Существование и единственность решения данной задачи обосновываются с помощью принципа неподвижной точки Банаха. Полученные результаты имеют важное значение для развития теории дробных дифференциальных уравнений.

Ключевые слова: дробно-диффузионное уравнение, производная Римана–Лиувилля, обратная задача, задача Коши, принцип неподвижной точки Банаха, существование, единственность, восстановление коэффициента.

INVERSE PROBLEM OF DETERMINING THE COEFFICIENT FOR A FRACTIONAL DIFFUSION EQUATION

Abstract. This paper investigates an inverse problem of determining a time-dependent coefficient for a fractional diffusion equation posed in an unbounded domain. The equation involves a time-fractional derivative in the sense of Riemann–Liouville. First, using known exact solutions corresponding to the case

$q(t) \equiv 0$, the existence and uniqueness of the solution to the Cauchy problem are analyzed when the coefficient is present. In addition, an inverse problem of reconstructing the unknown coefficient is

formulated using the additional condition $u(0,t)=g(t)$. The existence and uniqueness of the solution are established by applying the Banach fixed point principle. The results contribute to the theory of fractional differential equations.

Keywords: fractional diffusion equation, Riemann–Liouville derivative, inverse problem, Cauchy problem, Banach fixed point principle, existence, uniqueness, coefficient reconstruction.

So‘nggi yillarda kasr tartibli hosilalar qatnashgan differensial tenglamalar matematik fizika, muhandislik, biologiya va iqtisodiyot kabi ko‘plab sohalarda keng qo‘llanilmoqda. Bunday tenglamalar klassik diffuziya modellariga nisbatan murakkab xotira effektlarini hisobga olish imkonini beradi.

Kasr tartibli diffuziya tenglamalari anomal diffuziya jarayonlarini tasvirlashda muhim ahamiyatga ega bo'lib, ularning matematik nazariyasini o'rganish muhim masalalardan biri hisoblanadi. Xususan, bunday tenglamalar uchun boshlang'ich va chegaraviy masalalarning yechimlari, shuningdek, turli xil teskari masalalar keng tadqiq qilinmoqda.

Cheksiz sohada qaraladigan vaqt bo'yicha kasr tartibli diffuziya tenglamasi uchun Duhamel prinsipi asosida yechimlarni qurish masalasi M.B. Borikhanov va B.T. Torebek tomonidan o'rganilgan. Ularning ishida $q(t) \equiv 0$ hol uchun aniq yechim olingan.

Mazkur ishda esa vaqtga bog'liq $q(t)$ koeffitsiyent qatnashgan hol qaraladi. Avvalo, mos Koshi masalasining mavjudligi va yagonaligi tadqiq qilinadi. Bundan tashqari, qo'shimcha ma'lumot yordamida noma'lum koeffitsiyentni aniqlash teskari masalasi o'rganiladi.

Teskari masalani yechishda Banaxning qo'zg'almas nuqta prinsipi asosiy vosita sifatida qo'llaniladi.

Masalaning qo'yilishi. Cheksiz $\mathbb{R}$ sohada quyidagi kasr tartibli diffuziya tenglamasi uchun Koshi masalasini qaraymiz:

$$u_t - D_{0+,t}^\alpha u_{xx} + q(t)u(x,t) = f(x,t), \quad x \in \mathbb{R}, \quad 0 < t \leq T, \quad (1)$$

boshlang'ich shart bilan

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}. \quad (2)$$

Bu yerda $D_{0+,t}^\alpha$ vaqt bo'yicha Riman–Liuvill kasr tartibli hosilasini bildiradi, $\alpha \in (0,1)$.

Mazkur ishda $q(t)$ vaqtga bog'liq noma'lum koeffitsiyent mavjud bo'lgan hol qaraladi. Avvalo, yuqoridagi Koshi masalasi uchun yechimning mavjudligi va yagonaligi o'rganiladi.

Shuningdek, quyidagi qo'shimcha shart berilgan bo'lsin:

$$u(0,t) = g(t), \quad 0 < t \leq T. \quad (3)$$

Ushbu qo'shimcha ma'lumot yordamida $q(t)$ koeffitsiyentni aniqlash teskari masalasi qaraladi. Teskari masalaning yechimining mavjudligi va yagonaligi Banaxning qo'zg'almas nuqta prinsipi yordamida asoslanadi.

Ta'rif. (1) – (2) masalaning klassik yechimi deb shunday $u(x,t)$ funksiyaga aytiladiki, u quyidagi shartlarni qanoatlantiradi:

$$u, u_t, u_{xx}, D_{0+,t}^\alpha u_{xx} \in C_b(\Omega), \quad \Omega = \mathbb{R} \times (0, T],$$

ya'ni ushbu funksiyalar Ω sohada uzluksiz va chegaralangan.

1. $u(x,t)$ funksiya boshlang'ich shartni qanoatlantiradi

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}.$$

2. Ixtiyoriy $(x,t) \in \Omega$ uchun $u(x,t)$ funksiya (1) tenglamani qanoatlantiradi, ya'ni

$$u_t - D_{0+,t}^\alpha u_{xx} + q(t)u(x,t) = f(x,t).$$

To'g'ri masala. Dastlab ishda keltirilgan ayrim natijalardan foydalanamiz.

Lemma 1. Agar $q(t) \equiv 0$, $\varphi \in C_b(\mathbb{R})$ va $f(x,t) \in C_b(\Omega)$ bo'lsa, u holda (1) - (2) Koshi masalasining $u(x,t)$ yechimi va uning hosilalari uchun quyidagi baho o'rinli:

$$|u(x,t)| + |u_t(x,t)| + |D_{0+,t}^\alpha u_{xx}(x,t)| \leq C(\|\varphi\|_{C_b(\mathbb{R})} + \|f\|_{C_b(\Omega)}), \quad (4)$$

bu yerda $C > 0$ o'zgarmas son bo'lib, x va t ga bog'liq emas.

Endi (1) tenglamada $q(t)u(x,t)$ hadni tenglamaning o'ng tomoniga o'tkazamiz va ishda keltirilgan natijadan foydalanib (1)-(2) masalaning yechimi uchun mos integral tenglamani quyidagicha yozish mumkin:

$$u(x,t) = \Phi(x,t) - \int_0^t \int_{\mathbb{R}} G(x-\xi, t-\tau) q(\tau) u(\xi, \tau) d\xi d\tau. \quad (5)$$

Bu yerda

$$\Phi(x, t) = \int_{\mathbb{R}} G(x - \xi, t) \varphi(\xi) d\xi + \int_0^t \int_{\mathbb{R}} G(x - \xi, t - \tau) f(\xi, \tau) d\xi d\tau, \quad (6)$$

va $G(x, t)$ funksiya quyidagi ko'rinishda aniqlanadi:

$$G(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} E_{1-\alpha}(-\gamma^2 t^{1-\alpha}) d\xi, \quad (7)$$

bu yerda $E_{1-\alpha}(\cdot)$ Mittag-Leffler funksiyasini bildiradi.

Teorema 1. Agar $q(t) \in C[0, T]$, $\varphi(x) \in C_b(\mathbb{R})$ va $f(x, t) \in C_b(\Omega)$ bo'lsa, u holda (5) integral tenglamaning

$C(\Omega)$ sinfga tegishli yagona yechimi mavjud.

Isbot. Teoremani isbotlashda ketma-ket yaqinlashishlar usulidan foydalanamiz.

Yaqinlashishlar ketma-ketligi quyidagicha aniqlanadi:

$$u_m(x, t) = \int_0^t \int_{\mathbb{R}} G(x - \xi, t - \tau) q(\tau) u_{m-1}(\xi, \tau) d\xi d\tau, \quad m=1, 2, \dots \quad (8)$$

boshlang'ich yaqinlashish esa

$$u_0(x, t) = \Phi(x, t) \quad (9)$$

ko'rinishda olinadi.

Shunday qilib, ketma-ketlikning dastlabki hadlari quyidagi ko'rinishga ega:

$$u_1(x, t) = \int_0^t \int_{\mathbb{R}} G(x - \xi, t - \tau) q(\tau) \Phi(\xi, \tau) d\xi d\tau,$$

$$u_2(x, t) = \int_0^t \int_{\mathbb{R}} G(x - \xi, t - \tau) q(\tau) u_1(\xi, \tau) d\xi d\tau,$$

va hokazo.

Lemma 1 dan foydalanib $u_m(x, t)$ funksiyalar uchun quyidagi bahoni olish mumkin:

$$|u_m(x, t)| \leq C(\|\varphi\|_{C_b(\mathbb{R})} + \|f\|_{C_b(\Omega)}) \|q\|_{C[0, T]}^m \frac{t^m}{m!}, \quad m=0, 1, 2, \dots \quad (10)$$

Yuqorida keltirilgan baholardan foydalanib

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t)$$

qator Ω sohada tekis yaqinlashuvchi ekanligi kelib chiqadi. Shuning uchun $u(x, t)$ funksiya $C(\Omega)$ sinfga tegishli bo'ladi.

Bundan tashqari $u(x, t)$ funksiya uchun quyidagi baho o'rinni:

$$|u(x, t)| \leq C(\|\varphi\|_{C_b(\mathbb{R})} + \|f\|_{C_b(\Omega)}) \exp(\|q\|_{C[0, T]} t).$$

Aytaylik $u^*(x, t)$ funksiya (1) - (2) masalaning $q^*(t)$, $\varphi^*(x)$ va $f^*(x, t)$ funksiyalarga mos yechimi bo'lsin.

U holda

$$\bar{u}(x, t) = u(x, t) - u^*(x, t)$$

funksiyani qaraymiz.

Quyidagi belgilashlarni kiritamiz:

$$\bar{q}(t) = q(t) - q^*(t), \quad \bar{\varphi}(x) = \varphi(x) - \varphi^*(x), \quad \bar{f}(x, t) = f(x, t) - f^*(x, t).$$

U holda $\bar{u}(x, t)$ funksiya quyidagi integral tenglamani qanoatlantiradi:

$$\begin{aligned} \bar{u}(x,t) = & \bar{\Phi}(x,t) - \int_0^t \bar{q}(\tau) \int_{\mathbb{R}} G(x-\xi, t-\tau) u(\xi, \tau) d\xi d\tau \\ & - \int_0^t q^*(\tau) \int_{\mathbb{R}} G(x-\xi, t-\tau) \bar{u}(\xi, \tau) d\xi d\tau. \end{aligned} \quad (12)$$

Xuddi yuqorida keltirilganidek, ketma-ket yaqinlashishlar usulidan foydalansak, $\bar{u}(x,t)$ funksiya uchun quyidagi bahoni olish mumkin:

$$|\bar{u}(x,t)| \leq M(\|\bar{\varphi}\|_{C_b(\mathbb{R})} + \|\bar{q}\|_{C[0,T]} + \|\bar{f}\|_{C_b(\Omega)}), \quad (13)$$

bu yerda M yetarlicha katta musbat o'zgarmasdir.

$x=0$ ni (1) tenglamaga qo'yib va (3) qo'shimcha shartdan foydalanib, oddiy almashtirishlardan so'ng noma'lum $q(t)$ funksiyani aniqlash uchun quyidagi integral tenglamani hosil qilamiz:

$$q(t) = Q(t) + \int_0^t \int_0^s \int_{\mathbb{R}} K(\xi, s, \tau, t) q(\tau) u(\xi, \tau) d\xi d\tau ds, \quad (14)$$

bu yerda

$$K(\xi, s, \tau, t) = \frac{1}{g(t)} \frac{1}{\Gamma(1-\alpha)} G_{\xi, \xi}(\xi, s-\tau)(t-s)^{-\alpha},$$

$$Q(t) = f(0,t) - g'(t) + D_t^\alpha \left(\frac{\partial^2}{\partial x^2} u_0(x,t) \right) \Big|_{x=0}.$$

(14) integral tenglamaning yechimi $q(t)$ funksiyaga bog'liq bo'lib, $u=u(x,t,q)$ ko'rinishda yoziladi. Endi (14) tenglamaning o'ng tomoniga mos operatorni quyidagicha aniqlaymiz:

$$W[q](t) = Q(t) + \int_0^t \int_0^s \int_{\mathbb{R}} K(\xi, s, \tau, t) q(\tau) u(\xi, \tau) d\xi d\tau ds.$$

Shunda (14) tenglama quyidagi operator tenglama ko'rinishga ega bo'ladi:

$$q(t) = W[q](t). \quad (15)$$

$Q_0 = \max_{t \in [0,T]} |Q(t)|$ deb belgilaymiz va $\rho > 0$ sonni fiksatsiya qilamiz. Quyidagi sharni qaraymiz

$$B_T(Q, \rho) = \{q(t) \in C[0,T] : \|q - Q\| \leq \rho\}.$$

Teorema 2. Agar $f(x,t) \in C_b(\Omega_T)$, $\varphi(x) \in C_b(\mathbb{R})$, $g(t) \in C[0,T]$ va $|g(t)| \geq \frac{1}{g_0} > 0$ bo'lsa, bunda

$$g_0 < \frac{1}{M(\|\Phi\|_\sigma + \rho) + u_{00}},$$

hamda moslik sharti $g(0) = \varphi(0)$ bajarilsa, u holda shunday $T^* \in (0,T)$ mavjudki, (1) – (3) teskari masala $B_T(Q, \rho)$ Sharda yagona yechimga ega bo'ladi.

Avvalo W operator $T > 0$ yetarlicha kichik bo'lganda $B_T(Q, \rho)$ sharni o'ziga akslantirishini ko'rsatamiz, ya'ni ixtiyoriy $q(t) \in B_T(Q, \rho)$ uchun $W[q](t) \in B_T(Q, \rho)$ bo'ladi.

$q(t)$ funksiya uzluksiz bo'lgani sababli (15) formula bilan aniqlangan $W[q](t)$ ham uzluksiz bo'ladi. Bundan tashqari quyidagi baho o'rinli:

$$\|W[q] - Q\| \leq g_0(\|\varphi\|_{C_b(\mathbb{R})} + \|f\|_{C_b(\Omega)}) \|q\| \exp(\|q\|T).$$

Bu baho (10) tengsizlikdan kelib chiqadi. Agar $q(t) \in B_T(Q, \rho)$ bo'lsa, u holda

$$\|q\| \leq \|Q\| + \rho. \quad (16)$$

Shuni yuqoridagi bahoga qo'ysak,

$$\|W[q] - \Phi\| \leq g_0 (\|\varphi\|_{C_\beta(\mathbb{R})} + \|f\|_{C_\beta(\Omega)}) (\|Q\| + \rho) \exp((\|Q\| + \rho)T).$$

Endi

$$M_1(T) = g_0 (\|\varphi\|_{C_\beta(\mathbb{R})} + \|f\|_{C_\beta(\Omega)}) (\|Q\| + \rho) \exp((\|Q\| + \rho)T)$$

deb belgilaymiz. Agar $M_1(T) = \rho$ tenglamaning musbat ildizi T_1 bo'lsa, u holda $T \in [0, T_1]$ uchun

$W[q](t) \in B_T(\Phi, \rho)$ bo'ladi.

Endi $q(t)$ va $\hat{q}(t)$ funksiyalarni $B_T(\Phi, \rho)$ shardan olib, $\|W[q] - W[\hat{q}]\|_\sigma$ ni baholaymiz. Shunda

$$\|W[q] - W[\hat{q}]\| \leq g_0 (\|\bar{q}\| \|u\| + \|\hat{q}\| \|\bar{u}\|).$$

Mos baholardan foydalanib quyidagini olamiz:

$$\|W[q] - W[\hat{q}]\| \leq g_0 \|\bar{q}\| \exp(\|q\|T) (M\|\hat{q}\| + \|\varphi\|_{C_\beta(\mathbb{R})} + \|f\|_{C_\beta(\Omega)}).$$

(16) tengsizlikdan foydalanib

$$\|W[q] - W[\hat{q}]\| \leq g_0 \|\bar{q}\| \exp\left((\|\Phi\|_\sigma + \rho)T \frac{T^{1-\sigma}}{1-\sigma}\right) (M(\|Q\| + \rho) + \|\varphi\|_{C_\beta(\mathbb{R})} + \|f\|_{C_\beta(\Omega)}).$$

Endi

$$M_2(T) = g_0 \exp((\|Q\| + \rho)T) (M(\|Q\| + \rho) + \|\varphi\|_{C_\beta(\mathbb{R})} + \|f\|_{C_\beta(\Omega)})$$

deb belgilaymiz. Agar $M_2(T) = 1$ tenglamaning musbat ildizi T_2 bo'lsa, $T \in [0, T_2)$ uchun W operator siquvchi bo'ladi.

Shuning uchun $T^* = \min(T_1, T_2)$ tanlanganda W operator $B_T(Q, \rho)$ sharida siquvchi bo'ladi. Banaxning qo'zg'almas nuqta prinsipi asosida (15) tenglama yagona yechimga ega.

Teorema isbotlandi.

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STOCHASTIC PERTURBATIONS OF A NORMALIZED TENT-TYPE MAPS

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Abstract. We study stochastic perturbations of a rescaled tent-type map $\bar{T}_\omega$ on the unit interval $[0,1]$. For parameters $\sqrt[4]{2} < \omega < \sqrt{2}$, the corresponding deterministic system possesses two disjoint invariant intervals $Q_1(\omega)$ and $Q_2(\omega)$ which are permuted by the map $\bar{T}_\omega$ and their union form a chaotic attractor. We consider a nonlinear autoregressive stochastic process $X_n(\underline{\xi}, \varepsilon) := \{X_n(\underline{\xi}, \varepsilon), n \geq 0\}$, $\varepsilon > 0$, obtained by adding bounded noise and applying a reflection map that preserves the state space $[0,1]$. It is shown that for sufficiently small noise intensity ε , the perturbed system admits two disjoint invariant intervals $\tilde{Q}_1(\omega, \varepsilon)$ and $\tilde{Q}_2(\omega, \varepsilon)$ which form the support of a unique stationary probability measure. Explicit formulas for these intervals are derived, and it is proved that, as $\varepsilon \rightarrow 0$, the stochastic attractor converges to the deterministic chaotic attractor $Q_1(\omega) \cup Q_2(\omega)$.

Keywords. Tent map, chaos, nonlinear autoregressive process, invariant set.

NORMALLASHGAN CHODIRSIMON AKSLANTIRISHLARNING STOXAСТИK
SILJISHLARI

Annotatsiya. Ushbu ishda $[0,1]$ birlik kesmada aniqlangan normallashgan chodirsimon $\bar{T}_\omega$ akslantirishning stoxastik siljishlari o'rganiladi. $\sqrt[4]{2} < \omega < \sqrt{2}$ parametrlar uchun mos deterministik sistema ikkita o'zaro kesishmaydigan invariant interval $Q_1(\omega)$ va $Q_2(\omega)$ ga ega bo'lib, ular $\bar{T}_\omega$ akslantirish tomonidan o'zaro almashadi va ularning birlashmasi xotik attraktorni hosil qiladi. Biz chegaralangan shovqin qo'shish hamda holatlar fazosi $[0,1]$ ni saqlovchi akslantirish operatorini qo'llash orqali hosil qilingan noxiziqli avtoregressiv stoxastik jarayon $X(\underline{\xi}, \varepsilon) := \{X_n(\underline{\xi}, \varepsilon), n \geq 0\}$, $\varepsilon > 0$, ni qaraymiz. Ko'rsatiladiki, yetarlicha kichik ε shovqin intensivligi uchun siljirilgan sistema ikkita o'zaro kesishmaydigan invariant interval $\tilde{Q}_1(\omega, \varepsilon)$ va $\tilde{Q}_2(\omega, \varepsilon)$ ga ega bo'lib, ular yagona statsionar ehtimollik o'lchovining tayanchini (supportini) tashkil etadi. Mazkur intervallar uchun aniq formulalar keltiriladi va $\varepsilon \rightarrow 0$ da stoxastik attraktor deterministik xotik attraktor $Q_1(\omega) \cup Q_2(\omega)$ ga yaqinlashishi isbotlanadi.

Kalit so'zlar: chodirsimon akslantirish, xaos, noxiziqli avtoregressiv jarayon, invariant to'plam.

СТОХАСТИЧЕСКИЕ ВОЗМУЩЕНИЯ НОРМАЛИЗОВАННЫХ ПАЛАТКООБРАЗНЫХ
ОТОБРАЖЕНИЙ

Аннотация. В данной работе изучаются стохастические возмущения нормализованных палаткообразных отображений $\bar{T}_\omega$ на единичном отрезке $[0,1]$. При параметрах $\sqrt[4]{2} < \omega < \sqrt{2}$ соответствующая детерминированная система обладает двумя непересекающимися инвариантными интервалами $Q_1(\omega)$ и $Q_2(\omega)$, которые переставляются отображением $\bar{T}_\omega$, а их объединение образует хаотический аттрактор. Рассматривается нелинейный авторегрессионный стохастический процесс $X(\underline{\xi}, \varepsilon) := \{X_n(\underline{\xi}, \varepsilon), n \geq 0\}$, $\varepsilon > 0$, полученный добавлением ограниченного шума и применением отражающего отображения, сохраняющего фазовое пространство $[0,1]$. Показано, что при достаточно малой интенсивности шума ε возмущенная система допускает два непересекающихся инвариантных интервала $\tilde{Q}_1(\omega, \varepsilon)$ и $\tilde{Q}_2(\omega, \varepsilon)$,

которые образуют носитель единственной стационарной вероятностной меры. Получены явные формулы для этих интервалов и доказано, что при $\varepsilon \rightarrow 0$ стохастический аттрактор сходится к детерминированному хаотическому аттрактору $Q_1(\omega) \cup Q_2(\omega)$.

Ключевые слова: палаткообразных отображений, хаос, нелинейный авторегрессионный процесс, инвариантное множество.

Introduction. Piecewise linear interval maps provide a classical and important class of models in one-dimensional dynamics. Among them, tent maps occupy a central place due to their simple analytic form and rich dynamical behavior, ranging from stable motion to chaos (see [1], [2]). They arise naturally in symbolic dynamics, ergodic theory, and the study of invariant measures for expanding transformations (see for instance [1], [2], ...).

A general framework for the analysis of one-dimensional unimodal maps was developed by Milnor and Thurston [3], whose kneading theory shows that the global dynamics is essentially governed by the orbit of the critical point. Statistical and ergodic properties of piecewise expanding interval maps were studied in detail by Lasota and Mackey [4], while probabilistic aspects and invariant densities for piecewise linear transformations were further developed by Boyarsky and Góra [5].

In the present paper we consider a rescaled tent-type map $\bar{T}_\omega : [0,1] \rightarrow [0,1]$, associated with the classical tent family

$$T_\omega(x) = \begin{cases} \omega x, & x \leq \frac{1}{2}, \\ \omega(1-x), & x > \frac{1}{2}, \end{cases} \quad \omega > 0.$$

The dynamical behavior of this family strongly depends on the parameter ω . In particular, for $\omega \in (\sqrt[4]{2}, \sqrt{2})$, the deterministic system possesses two disjoint intervals $Q_1(\omega) = [0, \omega^2 - \omega]$, $Q_2(\omega) = [2 - \omega, 1]$,

which are permuted by $\bar{T}_\omega$ and whose union forms a chaotic attractor. Thus, although the map is not chaotic on the whole interval $[0,1]$, its long-term dynamics is completely determined by the chaotic behavior on the invariant set $Q(\omega) = Q_1(\omega) \cup Q_2(\omega)$.

Random perturbations of deterministic dynamical systems have been extensively studied in the literature. It is well known that sufficiently small noise may preserve essential statistical properties of the deterministic dynamics, while modifying the geometric structure of invariant sets and attractors (see for instance, Kifer [6]).

In the context of Feigenbaum-type unimodal systems, random perturbations were studied by E. Vul, Ya. Sinai and Khanin in [7] and Liu in [8], where a rigorous analysis of the invariant measure and its support was carried out, showing how the support changes as the noise intensity varies. Motivated by these developments, we study a stochastic perturbation of the map $\bar{T}_\omega$ given by the nonlinear autoregressive recursion

$$X_{n+1} = R(\bar{T}_\omega(X_n) + \varepsilon \xi_{n+1}),$$

where $\{\xi_n\}$ is a sequence of independent identically distributed random variables and $R: [0,1] \rightarrow [0,1]$ is a reflection map which keeps trajectories inside the unit interval. This model may be viewed as a bounded-noise perturbation of the deterministic tent dynamics, with reflection at the boundary preventing escape from the phase space. The main objective of the paper is to describe the invariant structure and the stationary behavior of the perturbed system for small values of the noise intensity ε . More precisely, we show that for $\omega \in (\sqrt[4]{2}, \sqrt{2})$ and ε is sufficiently small, the stochastic system admits two disjoint invariant intervals $\tilde{Q}_1(\omega, \varepsilon)$ and $\tilde{Q}_2(\omega, \varepsilon)$ which form the support of a unique stationary probability measure. We obtain explicit formulas for these intervals, prove that the stochastic dynamics alternates between the two components, and show that the corresponding stochastic attractor converges to the deterministic chaotic attractor $Q_1(\omega) \cup Q_2(\omega)$, as $\varepsilon \rightarrow 0$.

The paper is organized as follows. In Section 2 we introduce the rescaled tent map, establish the deterministic invariant structure, and formulate the main results for both the deterministic and stochastic models. In Section 3 we provide detailed proofs of the main theorems.

Preliminaries and Main Results. In this section we introduce the deterministic tent map, define the stochastic process, and formulate the main results of the paper.

1. The rescaled tent map. Let $\omega > 0$. The classical tent map [3] is defined as (see Figure 1)

$$T_{\omega}(x) = \begin{cases} \omega x, & 0 \leq x \leq \frac{1}{2}, \\ \omega(1-x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

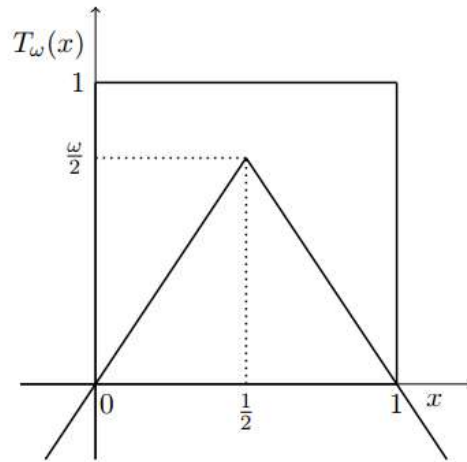


Figure 1

For $1 < \omega \leq 2$, the interval $I_{\omega} = \left[\omega - \frac{\omega^2}{2}, \frac{\omega}{2} \right]$ is forward invariant. By rescaling I_{ω} onto $[0, 1]$, we obtain the rescaling tent map $\bar{T}_{\omega} : [0, 1] \rightarrow [0, 1]$, given by (see Figure 2)

$$\bar{T}_{\omega}(x) = \begin{cases} 2 - \omega + \omega x, & 0 \leq x \leq c_{\omega} := \frac{\omega-1}{\omega}, \\ \omega(1-x), & c_{\omega} \leq x \leq 1. \end{cases}$$

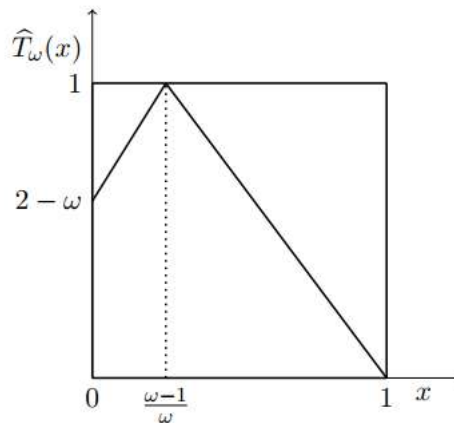


Figure 2

The point c_{ω} is the critical point of the map and satisfies $\bar{T}_{\omega}(c) = 1$. The dynamical behavior of $\bar{T}_{\omega}$ depends strongly on the parameter ω . In the case, $\omega \in (\sqrt[4]{2}, \sqrt{2})$, the dynamics is concentrated on two invariant components.

2. Invariant intervals and deterministic attractor. Define two intervals as

$$Q_1(\omega) = [0, \omega^2 - \omega], \quad Q_2(\omega) = [2 - \omega, 1].$$



These intervals are disjoint and satisfy $\bar{T}_\omega(Q_1(\omega)) = Q_2(\omega)$, $\bar{T}_\omega(Q_2(\omega)) = Q_1(\omega)$.

Hence their union $Q(\omega) = Q_1(\omega) \cup Q_2(\omega)$ is forward invariant.

Definition 2.1 (see [9]). A compact set $Q \subset [0,1]$ is called an attractor if

- $\bar{T}_\omega(Q) = Q$;
- there exists a set $B(Q) \subset [0,1]$ of positive Lebesgue measure such that for every $x \in B(Q)$ there exists $N := N(x) \geq 1$ with $\bar{T}_\omega^n(x) \in Q$ for all $n \geq N$.

According to Theorem 1 in [10], the restriction $(T_\omega^2)|_{Q_1(\omega)}$ is conjugate to the tent map T_{ω^2} . Therefore, the second iterate of the map on $Q_1(\omega)$ inherits the dynamical properties of T_{ω^2} . In particular, since $\sqrt[4]{2} < \omega < \sqrt{2}$, we have $\omega^2 > \sqrt{2}$, and it is well known that the tent map T_{ω^2} is chaotic in the sense of Devaney[11]. Hence, by conjugacy, the map $(T_\omega^2)|_{Q_1(\omega)}$ is also chaotic.

Definition 2.2. Let $Q \subset [0,1]$ be an invariant set. The first entry set into Q is defined by

$$E = \bar{T}_\omega^{-1}(Q) = \{x \in [0,1] : \bar{T}_\omega(x) \in Q\}.$$

These sets describe how orbits from the complement of Q enter the attractor.

Theorem 2.1. Consider tent family T_ω , $\sqrt[4]{2} < \omega < \sqrt{2}$ and its rescaled $\bar{T}_\omega$. Then the following statements hold.

1. The intervals $Q_1(\omega)$ and $Q_2(\omega)$ are disjoint and are permuted by $\bar{T}_\omega$;
2. The set $Q_1(\omega) \cup Q_2(\omega)$ is a chaotic attractor.
3. We put $E_1(\omega) := \left(\omega^2 - \omega, \frac{2(\omega-1)}{\omega}\right)$. Then $\bar{T}_\omega(E_1(\omega)) \subset Q_2(\omega)$, and
4. $\text{dist}(Q_1(\omega), Q_2(\omega)) = 2 - \omega^2$.

3. Stochastic perturbations. To describe random effects, we consider the stochastic recursion

$$X_{n+1} = R(\bar{T}_\omega(X_n) + \varepsilon \xi_{n+1}),$$

where $\{\xi_n\}$ are independent identically distributed random variables (i.i.d.) on $[-1,1]$, and $\varepsilon > 0$ is the noise intensity.

The reflection map $R: \mathbb{P} \rightarrow [0,1]$ is defined by

$$R(y) = \begin{cases} -y, & y < 0, \\ y, & 0 \leq y \leq 1, \\ 2 - y, & y > 1. \end{cases}$$

This construction ensures that the process remains in $[0,1]$.

The process $\{X_n\}$ is a time-homogeneous Markov chain with transition kernel

$$P(x, A) = \mathbb{P}(X_{n+1} \in A | X_n = x).$$

Definition 2.3 (see [11]). A probability measure π on $[0,1]$ is called stationary if

$$\pi(A) = \int_{[0,1]} P(x, A) d\pi(x),$$

for all Borel sets $A \subset [0,1]$.

4. Invariant structure of the stochastic system. Define the enlarged intervals

$$\tilde{Q}_1(\omega, \varepsilon) := [0, \omega^2 - \omega + (\omega + 1)\varepsilon], \quad \tilde{Q}_2(\omega, \varepsilon) := [2 - \omega - \varepsilon, 1].$$

These sets represent stochastic neighborhoods of the deterministic invariant intervals $Q_1(\omega)$ and $Q_2(\omega)$. The enlargement reflects the effect of random perturbations, which allow trajectories to deviate from the deterministic images by at most ε at each iteration, followed by the reflection map that keeps the dynamics inside the unit interval.

More precisely, under the stochastic recursion

$$X_{n+1} = R(\bar{T}_\omega(X_n) + \varepsilon \xi_{n+1}),$$

the image of a point in $Q_1(\omega)$ is no longer confined exactly to $Q_2(\omega)$, but is spread over an interval of width proportional to ε . This motivates the definition of $\tilde{Q}_2(\omega, \varepsilon)$ as a natural enlargement of $Q_2(\omega)$. Similarly, the image of $Q_2(\omega)$ under the stochastic dynamics spreads into $\tilde{Q}_1(\omega, \varepsilon)$. In this sense, the sets $\tilde{Q}_1(\omega, \varepsilon)$ and $\tilde{Q}_2(\omega, \varepsilon)$ can be viewed as the minimal deterministic intervals that contain the one-step stochastic images of $Q_2(\omega)$ and $Q_1(\omega)$, respectively. As $\varepsilon \rightarrow 0$, these enlarged intervals converge to the deterministic invariant components, $\tilde{Q}_1(\omega, \varepsilon) \rightarrow Q_1(\omega)$, $\tilde{Q}_2(\omega, \varepsilon) \rightarrow Q_2(\omega)$. Thus, they provide a natural description of the invariant geometric structure of the stochastic system and will play a central role in the analysis of its long-term behavior. We set $\varepsilon_0(\omega) := \frac{(\omega-1)(2-\omega^2)}{\omega(\omega+1)}$. Next we formulate the main theorem on support of stationary measure of random process $\Xi(\omega, \varepsilon)$.

Theorem 2.2. $\bar{T}_\omega$ is rescaled map, $\sqrt[4]{2} < \omega < \sqrt{2}$ and assume $0 < \varepsilon < \varepsilon_0(\omega)$. Let $X_n(\omega, \varepsilon)$ is defined by (2) and $\{\xi_n\}$ is i.i.d with uniform random variables on $[0,1]$. Then there exist two closed intervals $\tilde{Q}_1(\omega, \varepsilon)$ and $\tilde{Q}_2(\omega, \varepsilon)$ such that the following properties hold.

1. The set $\Lambda_\varepsilon = \tilde{Q}_1(\omega, \varepsilon) \cup \tilde{Q}_2(\omega, \varepsilon)$ is forward invariant for the stochastic dynamics.
2. The intervals are explicitly given by

$$\tilde{Q}_1(\omega, \varepsilon) = [0, \omega^2 - \omega + (\omega+1)\varepsilon], \quad \tilde{Q}_2(\omega, \varepsilon) = [2 - \omega - \varepsilon, 1].$$

3. The dynamics alternates between the two components in the sense that

$$X_n(\omega, \varepsilon) \in \tilde{Q}_1(\omega, \varepsilon) \Rightarrow X_{n+1}(\omega, \varepsilon) \in \tilde{Q}_2(\omega, \varepsilon),$$

and

$$X_n(\omega, \varepsilon) \in \tilde{Q}_2(\omega, \varepsilon) \Rightarrow X_{n+1}(\omega, \varepsilon) \in \tilde{Q}_1(\omega, \varepsilon).$$

4. There exists a unique stationary probability measure π_ε for the stochastic process $\{X_n(\omega, \varepsilon)\}$, whose support satisfies $\text{supp}(\pi_\varepsilon) = \Lambda_\varepsilon$.
5. The stationary measure distributes equally between the two components:

$$\pi_\varepsilon(\tilde{Q}_1(\omega, \varepsilon)) = \pi_\varepsilon(\tilde{Q}_2(\omega, \varepsilon)) = \frac{1}{2}.$$

3. Proofs of the Main Results.

Proof of Theorem 2.1

Proof. Let

$$\bar{T}_\omega(x) = \begin{cases} 2 - \omega + \omega x, & 0 \leq x \leq c, \\ \omega(1 - x), & c \leq x \leq 1, \end{cases} \quad c = \frac{\omega-1}{\omega},$$

where $\sqrt[4]{2} < \omega < \sqrt{2}$. We prove each statement separately.

(1) The intervals $Q_1(\omega)$ and $Q_2(\omega)$ are disjoint and are permuted by $\bar{T}_\omega$. Recall that

$$Q_1(\omega) = [0, \omega^2 - \omega], \quad Q_2(\omega) = [2 - \omega, 1].$$

Since $\omega < \sqrt{2}$, we have $\omega^2 < 2$, hence $\omega^2 - \omega < 2 - \omega$. Therefore, $Q_1(\omega) \cap Q_2(\omega) = \emptyset$, so the intervals are disjoint. We now compute their images. First note that $\omega^2 - \omega - c = \omega^2 - \omega - \frac{\omega-1}{\omega} = \frac{(\omega-1)^2(\omega+1)}{\omega} > 0$.

Thus $c \in Q_1(\omega)$, so $Q_1(\omega)$ intersects both monotone branches of $\bar{T}_\omega$. Since $\bar{T}_\omega$ is increasing on $[0, c]$ and decreasing on $[c, 1]$, we have $\bar{T}_\omega(Q_1(\omega)) = [\bar{T}_\omega(0), \bar{T}_\omega(c)]$. Using $\bar{T}_\omega(0) = 2 - \omega$, $\bar{T}_\omega(c) = 1$, we get

$\bar{T}_\omega(Q_1(\omega)) = [2 - \omega, 1] = Q_2(\omega)$. Next, since $2 - \omega - c = 2 - \omega - \frac{\omega-1}{\omega} = \frac{(2-\omega)(\omega-1)}{\omega} > 0$, the whole interval $Q_2(\omega)$ lies to the right of c , hence on the decreasing branch. Therefore $\bar{T}_\omega(Q_2(\omega)) = [\bar{T}_\omega(1), \bar{T}_\omega(2 - \omega)]$.

It is clear that $\bar{T}_\omega(1) = 0$, $\bar{T}_\omega(2 - \omega) = \omega(1 - (2 - \omega)) = \omega(\omega - 1) = \omega^2 - \omega$. Then, $\bar{T}_\omega(Q_2(\omega)) = [0, \omega^2 - \omega] = Q_1(\omega)$. Consequently, $\bar{T}_\omega(Q_1(\omega)) = Q_2(\omega)$, $\bar{T}_\omega(Q_2(\omega)) = Q_1(\omega)$, and applying $\bar{T}_\omega$ once more gives $\bar{T}_\omega^2(Q_1(\omega)) = Q_1(\omega)$, $\bar{T}_\omega^2(Q_2(\omega)) = Q_2(\omega)$.

(2.1). The set $Q(\omega) = Q_1(\omega) \cup Q_2(\omega)$ is chaotic. By the rescaled result for the tent family, the restriction $\bar{T}_\omega^2|_{Q_1(\omega)}$ is conjugate to the rescaled tent map with parameter ω^2 . Since $\sqrt[4]{2} < \omega < \sqrt{2}$, we have $\sqrt{2} < \omega^2 < 2$. Therefore the tent map with parameter ω^2 is chaotic, and hence so is $\bar{T}_\omega^2|_{Q_1(\omega)}$. Because $\bar{T}_\omega$ permutes $Q_1(\omega)$ and $Q_2(\omega)$, the dynamics on $Q_2(\omega)$ is obtained from that on $Q_1(\omega)$ after one iteration. It follows that the restriction $\bar{T}_\omega|_{Q(\omega)}$ is chaotic as well.

(2.2), Every point outside $Q(\omega)$, except for the unique fixed point $x^*(\omega) = \frac{\omega}{\omega+1}$, eventually enters $Q(\omega)$. It remains to show that $\forall x \in [0, 1] \setminus Q(\omega) \exists n \geq 1$ such that $\bar{T}_\omega^n(x) \in Q(\omega)$. It is obvious that $[0, 1] \setminus Q(\omega) = (\omega^2 - \omega, 2 - \omega)$. Let $G = (\omega^2 - \omega, 2 - \omega)$ denote this gap. Consider first the interval

$$E_1(\omega) = \left(\omega^2 - \omega, \frac{2(\omega-1)}{\omega} \right).$$

By part (4), every point of E_1 is mapped directly into $Q_2(\omega)$. The remaining part of the gap is

$$J = \left[\frac{2(\omega-1)}{\omega}, 2 - \omega \right).$$

For $x \in J$, we have $x \geq c$, hence $\bar{T}_\omega(x) = \omega(1 - x)$. Since $x < 2 - \omega$, it follows that $\bar{T}_\omega(x) > \omega(1 - (2 - \omega)) = \omega(\omega - 1) = \omega^2 - \omega$, while from $x \geq \frac{2(\omega-1)}{\omega}$ we obtain $\bar{T}_\omega(x) \leq \omega \left(1 - \frac{2(\omega-1)}{\omega} \right) = 2 - \omega$. Thus $\bar{T}_\omega(J) \subset G$. Moreover, because the slope of $\bar{T}_\omega$ on J equals $-\omega$ with $\omega > 1$, the map is expanding there. Hence repeated iteration of any point in J , except possibly a repelling fixed point of the first return dynamics, eventually leaves J and enters E_1 , and therefore reaches $Q_2(\omega)$. Consequently every point of the gap eventually enters $Q(\omega)$. Thus $Q(\omega)$ is a chaotic attractor.

(3) The interval E_1 is a first entry interval into $Q_2(\omega)$. Let $E_1 = \left(\omega^2 - \omega, \frac{2(\omega-1)}{\omega} \right)$. We show that $\bar{T}_\omega(E_1) \subset Q_2(\omega) = [2 - \omega, 1]$. We distinguish two cases.

Case 1: $x \in (\omega^2 - \omega, c]$. Then $\bar{T}_\omega(x) = 2 - \omega + \omega x$. Since this branch is increasing, $\bar{T}_\omega(x) > 2 - \omega + \omega(\omega^2 - \omega) > 2 - \omega$, and $\bar{T}_\omega(x) \leq \bar{T}_\omega(c) = 1$. Hence $2 - \omega < \bar{T}_\omega(x) \leq 1$.

Case 2. Assume that $x \in [c, \frac{2(\omega-1)}{\omega}]$. Then $\bar{T}_\omega(x) = \omega(1-x)$. Since this branch is decreasing, $\bar{T}_\omega(x) \leq \bar{T}_\omega(c) = 1$. Also, $x < \frac{2(\omega-1)}{\omega} \Rightarrow 1-x > \frac{2-\omega}{\omega}$, and therefore $\bar{T}_\omega(x) = \omega(1-x) > 2-\omega$. So again $2-\omega < \bar{T}_\omega(x) \leq 1$. Combining the two cases, we obtain $\bar{T}_\omega(E_1(\omega)) \subset (2-\omega, 1] \subset Q_2(\omega)$. Thus $E_1(\omega)$ is a first entry interval into $Q_2(\omega)$.

(4) Distance between the components. Since $Q_1(\omega)$ and $Q_2(\omega)$ are disjoint compact intervals, their distance equals the difference between the left endpoint of $Q_2(\omega)$ and the right endpoint of $Q_1(\omega)$, namely

$\text{dist}(Q_1(\omega), Q_2(\omega)) = (2-\omega) - (\omega^2 - \omega) = 2 - \omega^2$. Since $\omega \rightarrow \sqrt{2}^-$ implies $\omega^2 \rightarrow 2^-$, it follows that $\text{dist}(Q_1(\omega), Q_2(\omega)) \rightarrow 0$.

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Recall that $\Lambda_\varepsilon = \tilde{Q}_1(\omega, \varepsilon) \cup \tilde{Q}_2(\omega, \varepsilon)$, where $\tilde{Q}_1(\omega, \varepsilon) = [0, \omega^2 - \omega + (\omega + 1)\varepsilon]$, $\tilde{Q}_2(\omega, \varepsilon) = [2 - \omega - \varepsilon, 1]$. As $\varepsilon \rightarrow 0$, the endpoints of these intervals converge to the endpoints of the deterministic invariant components:

$\omega^2 - \omega + (\omega + 1)\varepsilon \rightarrow \omega^2 - \omega$, $2 - \omega - \varepsilon \rightarrow 2 - \omega$. Hence $\tilde{Q}_1(\omega, \varepsilon) \rightarrow Q_1(\omega)$, $\tilde{Q}_2(\omega, \varepsilon) \rightarrow Q_2(\omega)$ in the Hausdorff metric. Therefore $\Lambda_\varepsilon \rightarrow Q_1(\omega) \cup Q_2(\omega)$ as $\varepsilon \rightarrow 0$. This proves the corollary.

Proof. Let $q := \omega^2 - \omega$, $r := 2 - \omega$, $\delta := r - q = 2 - \omega^2$, and consider the stochastic process

$$X_{n+1} = R(\bar{T}_\omega(X_n) + \varepsilon \xi_{n+1}), \quad \xi_n \square U[-1, 1]. \text{ Define}$$

$$\tilde{Q}_1(\omega, \varepsilon) = [0, \alpha], \quad \tilde{Q}_2(\omega, \varepsilon) = [\beta, 1],$$

$$\text{where } \alpha = q + (\omega + 1)\varepsilon, \quad \beta = r - \varepsilon.$$

(1) Forward invariance. Let $\Lambda_\varepsilon = \tilde{Q}_1(\omega, \varepsilon) \cup \tilde{Q}_2(\omega, \varepsilon)$. We show that $P(x, \Lambda_\varepsilon) = 1$ for all $x \in \Lambda_\varepsilon$.

(a) If $x \in \tilde{Q}_1(\omega, \varepsilon)$, then as shown below in part (3), $X_{n+1}(\omega, \varepsilon) \in \tilde{Q}_2(\omega, \varepsilon)$.

(b) If $x \in \tilde{Q}_2(\omega, \varepsilon)$, then as shown below, $X_{n+1}(\omega, \varepsilon) \in \tilde{Q}_1(\omega, \varepsilon)$. Thus $X_{n+1} \in \Lambda_\varepsilon$ almost surely, and Λ_ε is forward invariant.

(2) Explicit form of the intervals. By definition, $\tilde{Q}_1(\omega, \varepsilon) = [0, \omega^2 - \omega + (\omega + 1)\varepsilon]$, $\tilde{Q}_2(\omega, \varepsilon) = [2 - \omega - \varepsilon, 1]$. Moreover, we verify that these intervals are disjoint. Indeed,

$$\alpha < \beta \Leftrightarrow (\omega + 2)\varepsilon < \delta. \text{ Since } \varepsilon < \frac{(\omega - 1)\delta}{\omega(\omega + 1)} \text{ and } (\omega + 2)(\omega - 1) < \omega(\omega + 1), \text{ we obtain}$$

$$(\omega + 2)\varepsilon < \delta, \text{ hence } \tilde{Q}_1(\omega, \varepsilon) \cap \tilde{Q}_2(\omega, \varepsilon) = \emptyset.$$

(3) Alternation of the dynamics. **(a)** Let $x \in \tilde{Q}_1(\omega, \varepsilon)$. Define $b := 2 - \frac{2}{\omega}$. Then $\alpha < b$, hence $\tilde{Q}_1 \subset [0, b]$. Using monotonicity of $\bar{T}_\omega$, $\bar{T}_\omega([0, b]) = [r, 1]$. Therefore $\bar{T}_\omega(\tilde{Q}_1) \subset [r, 1]$. After adding noise and applying R , $R(z + \varepsilon \xi) \in [r - \varepsilon, 1] = \tilde{Q}_2$. Hence $X_n \in \tilde{Q}_1(\omega, \varepsilon) \Rightarrow X_{n+1} \in \tilde{Q}_2(\omega, \varepsilon)$.

(b) Let $x \in \tilde{Q}_2(\omega, \varepsilon) = [\beta, 1]$. Then $\bar{T}_\omega(x) = \omega(1 - x)$, and $\bar{T}_\omega(\tilde{Q}_2) = [0, q + \omega\varepsilon]$. After adding noise and applying R , $R(z + \varepsilon \xi) \in [0, q + (\omega + 1)\varepsilon] = \tilde{Q}_1$. Hence $X_n \in \tilde{Q}_2(\omega, \varepsilon) \Rightarrow X_{n+1} \in \tilde{Q}_1(\omega, \varepsilon)$.

(4) Existence, uniqueness and support of the stationary measure. Since Λ_ε is compact and the transition kernel is continuous, the Markov operator is Feller. By the Krylov–Bogolyubov theorem, there exists a stationary probability measure π_ε . From the alternation and the smoothing effect of the noise, the chain is irreducible on Λ_ε and strong Feller. Hence the stationary measure is unique. Moreover, every nonempty open subset of Λ_ε has positive probability to be reached in finitely many steps, which implies $\text{supp}(\pi_\varepsilon) = \Lambda_\varepsilon$.

(5) Equal distribution between the components. Let π_ε be stationary. Then

$$\pi_\varepsilon(\tilde{Q}_1) = \int P(x, \tilde{Q}_1) d\pi_\varepsilon(x).$$

By alternation, $P(x, \tilde{Q}_1) = 0$ on $\tilde{Q}_1$, $P(x, \tilde{Q}_1) = 1$ on $\tilde{Q}_2$. Hence $\pi_\varepsilon(\tilde{Q}_1) = \pi_\varepsilon(\tilde{Q}_2)$. Since the total mass is 1, we obtain $\pi_\varepsilon(\tilde{Q}_1) = \pi_\varepsilon(\tilde{Q}_2) = \frac{1}{2}$.

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AN APPROACH TO COMPUTING OPTIMAL COEFFICIENTS OF AN INTERPOLATION SPLINE

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Abstract. This paper addresses the mathematical challenge of constructing optimal interpolation splines within the Hilbert space. The research focuses on a variational approach where the spline is defined as a function minimizing a specific semi-norm functional. The study provides a structured analysis of the interpolation problem for both even and odd natural orders of m , ensuring that the resulting functions satisfy essential boundary conditions and maintain high-order smoothness across the partition of the interval $[0,1]$. This framework serves as a theoretical basis for developing efficient approximation algorithms.

Key words: Hilbert space, an interpolation spline, optimal coefficients, Sobolev's method.

ПОДХОД К ВЫЧИСЛЕНИЮ ОПТИМАЛЬНЫХ КОЭФФИЦИЕНТОВ ИНТЕРПОЛЯЦИОННОГО СПЛАЙНА

Аннотация. В данной работе рассматривается математическая задача построения оптимальных интерполяционных сплайнов в гильбертовом пространстве. Исследование сосредоточено на вариационном подходе, где сплайн определяется как функция, минимизирующая определенный функционал полу-нормы. В работе представлен структурированный анализ проблемы интерполяции как для четных, так и для нечетных натуральных порядков m , гарантирующий, что результирующие функции удовлетворяют существенным граничным условиям и сохраняют гладкость высокого порядка на разбиении интервала $[0,1]$. Эта структура служит теоретической основой для разработки эффективных алгоритмов аппроксимации.

Ключевые слова: Гильбертово пространство, интерполяционный сплайн, оптимальные коэффициенты, метод Соболева.

IKKI FAZALI MUHITDA YUZAGA KELADIGAN TENGLAMA UCHUN IKKI QATLAMLI POTENSIALLAR

Annotatsiya. Ushbu maqola Gilbert fazosida optimal interpolyatsion splaynlarini qurishning matematik masalasini ko'rib chiqadi. Tadqiqot variatsion yondashuvga qaratilgan bo'lib, unda splayn ma'lum bir yarim normaning funksionalini minimallashtiruvchi funksiya sifatida qaraladi. Tadqiqot m ning juft va toq natural tartiblari hollari uchun interpolyatsiya masalasining strukturaviy tahlilini taqdim etadi, natijada hosil bo'lgan funksiyalar muhim chegara shartlarini qondirishini va $[0,1]$ intervalining bo'linishi bo'yicha yuqori tartibli silliqlikni saqlab qolishini ta'minlaydi. Ushbu sistema samarali yaqinlashish algoritmlarini ishlab chiqish uchun nazariy asos bo'lib xizmat qiladi.

Kalit so'zlar: gilbert fazosi, interpolyatsion splayn, optimal koeffitsiyentlar, Sobolev usuli.

Spline functions are one of the key instruments of modern computational mathematics, finding wide application in function approximation, data interpolation, numerical modeling and the solution of applied problems. Splines are functions that partition a given interval into smaller segments, where the approximation is performed using low-degree polynomials (typically of the first, second, or third degree), thereby ensuring smoothness in the transitions between segments. The primary advantage of splines lies in their high flexibility, computational efficiency, and the ability to accurately model complex functions while maintaining the continuity of derivatives. These properties have rendered splines an indispensable tool in fields such as computer graphics, signal processing, medical visualization, engineering design, forecasting economics and data analysis.

I. Schoenberg's [1, 2] works on interpolating splines and their properties became the starting point for subsequent research. Among various types of splines, cubic splines have gained particular popularity, as they provide an optimal balance between smoothness (continuity of the first and second derivatives) and

computational efficiency. Significant contributions to the development of cubic splines were made by scientists such as D. Holladay and Carl de Boor [3–5].

Modern research in the field of splines continues to expand their capabilities. For instance, the works of L. Schumaker [6] on adaptive splines and their application in data approximation have made a significant contribution to the development of smoothing methods. It is also worth noting the research of G.V. Milovanovic and his colleagues, dedicated to splines in Sobolev spaces and their application in numerical methods [7]. Spline functions find application in a diverse range of disciplines. For example, in computer graphics, they are used to construct smooth curves and surfaces [8]; in signal processing—for data smoothing and signal reconstruction [9]; and in medical visualization—for enhancing image quality in computed tomography (CT) [10]. Furthermore, we present some results obtained in the theory of L-splines. Early contributions to L-spline theory include the works of Greville, Ahlberg, Nilson, and Walsh, as well as Schultz and Varga (see [6], p. 459). These works focused on natural L-splines, which emerge as solutions to the corresponding best interpolation problems and they were the first to investigate the order of approximation of generalized splines.

Statement of the problem. Suppose we are given the values y_β , $\beta = 0, 1, \dots, N$ at the points $x_\beta \in [0, 1]$, $\beta = 0, 1, \dots, N$. Consider the following interpolation problem.

Problem 1. Find the function $\varphi(x)$ in the space $W_2^{(m,0)}(0,1)$ satisfying the conditions:

$$\varphi(x_\beta) = y_\beta, \quad \beta = 0, 1, \dots, N \quad (1)$$

specifically, find the function $S_m(x)$ that minimizes the semi-norm

$$\|\varphi\|_{W_2^{(m,0)}} = \left(\int_0^1 (\varphi^{(m)}(x) - \varphi(x))^2 dx \right)^{1/2}, \quad \text{where } x_\beta \in [0, 1] - \text{the interpolation nodes.}$$

The solution $S_m(x)$ of problem 1 is a generalized spline and is uniquely determined with respect to the partition $\Delta: 0 = x_0 < x_1 < \dots < x_N = 1$ on the interval $[0, 1]$ as follows:

$S_m(x)$ - is a linear combination of exponential-trigonometric functions satisfying the identity $L * LS_m = 0$ on each sub-interval $(x_\beta, x_{\beta+1})$, $\beta = 0, 1, \dots, N-1$ of the partition, where L^* —adjoint operator L ;

$S_m(x)$ - is a linear combination of exponential-trigonometric functions satisfying the identity $LS_m = 0$ on the intervals $(-\infty; 0)$ и $(1; \infty)$;

$$S_m^{(\alpha)}(x_\beta^-) = S_m^{(\alpha)}(x_\beta^+), \quad \alpha = 0, 1, \dots, m \quad \text{and} \quad \beta = 1, 2, \dots, N-1;$$

$$S_m(x_\beta) = y_\beta, \quad \beta = 0, 1, \dots, N;$$

$S_m(x)$ satisfies the following boundary conditions:

$$S_m^{(m+i)}(1) + S_m^{(i)}(1) = 0, \quad i = 0, 1, \dots, m-2,$$

$$S_m^{(m+i)}(0) + S_m^{(i)}(0) = 0, \quad i = 0, 1, \dots, m-2.$$

Let m odd; then we shall consider the fundamental solution

$$G_m(x) = \frac{\text{sign} x}{2m} \cdot \left[\sinh(x) + \sum_{k=1}^{m-1} e^{x \cdot \cos\left(\frac{\pi k}{m}\right)} \cdot \cos\left(x \sin\left(\frac{\pi k}{m}\right) + \frac{\pi k}{m}\right) \right] \quad (2)$$

of the differential operator $\frac{d^{2m}}{dx^{2m}} - 1$ for odd natural numbers m , i.e., the solution to the equation

$$G_m^{(2m)}(x) - G_m(x) = \delta(x), \quad (3)$$

where δ – the Dirac's delta function. Then, we obtain

$$S_m(x) = \sum_{\gamma=0}^N C_\gamma G_m(x - x_\gamma) + Y_m(x), \quad (4)$$

where C_γ , $\gamma = 0, 1, \dots, N$, $G_m(x)$ is defined in (2), and $Y_m(x)$ is defined in

$$Y_m(x) = d_0 e^x + \sum_{k=1}^{\frac{m-1}{2}} e^{x \cos \frac{2\pi k}{m}} \left[d_{1,k} \cos \left(x \sin \left(\frac{2\pi k}{m} \right) \right) + d_{2,k} \sin \left(x \sin \left(\frac{2\pi k}{m} \right) \right) \right].$$

It is known that the solution $S_m(x)$ of the form (2) to Problem 1 exists and is unique for $N+1 \geq m$ and the coefficients C_γ , d_0 , d_{1k} and $d_{2k} \left(k = 1, 2, \dots, \frac{m-1}{2} \right)$ of the spline $S_m(x)$ are determined by the following system of $N+1+m$ linear equations:

$$\sum_{\gamma=0}^N C_\gamma G_m(x_\beta - x_\gamma) + Y_m(x_\beta) = \varphi(x_\beta), \quad \beta = \overline{0, N}, \quad (5)$$

$$\sum_{\gamma=0}^N C_\gamma e^{x_\gamma} = 0, \quad (6)$$

$$\sum_{\gamma=0}^N C_\gamma e^{x_\gamma \cos \frac{2\pi k}{m}} \cos \left(x_\gamma \sin \frac{2\pi k}{m} \right) = 0, \quad k = 1, \frac{m-1}{2}, \quad (7)$$

$$\sum_{\gamma=0}^N C_\gamma e^{x_\gamma \cos \frac{2\pi k}{m}} \sin \left(x_\gamma \sin \frac{2\pi k}{m} \right) = 0, \quad k = 1, \frac{m-1}{2}. \quad (8)$$

It is easy to show that the spline $S_m(x)$, defined by equation (4) with coefficients C_γ , d_0 , d_{1k} and $d_{2k} \left(k = 1, 2, \dots, \frac{m-1}{2} \right)$, satisfies conditions (i)-(v).

Next, we assume that m is an even natural number; then we will consider the fundamental solution:

$$G_m(x) = \frac{\operatorname{sign} x}{2m^2} \cdot \sum_{k=1}^m \left[(1-m) e^{x \cos \frac{2\pi k}{m}} \cos \left(x \sin \left(\frac{2\pi k}{m} \right) + \frac{2\pi k}{m} \right) + \right. \\ \left. + x e^{x \cos \frac{2\pi k}{m}} \cos \left(x \sin \left(\frac{2\pi k}{m} \right) + \frac{4\pi k}{m} \right) \right] \quad (9)$$

of the differential operator $\frac{d^{2m}}{dx^{2m}} - 2 \frac{d^m}{dx^m} + 1$ for odd natural numbers m , i.e., the solution to the equation:

$$G_m^{(2m)}(x) - 2G_m^{(m)}(x) + G_m(x) = \delta(x) \quad (10)$$

where δ – is the Dirac delta function. Then, we obtain

$$S_m(x) = \sum_{\gamma=0}^N C_\gamma G_m(x - x_\gamma) + Y_m(x), \quad (11)$$

where C_γ , $\gamma = 0, 1, \dots, N$, $G_m(x)$ is defined in (9) and

$$Y_m(x) = r_1 e^x + r_2 e^{-x} + \sum_{k=1}^{\frac{m-1}{2}} e^{x \cos \frac{2\pi k}{m}} \left[r_{1,k} \cos \left(x \sin \left(\frac{2\pi k}{m} \right) \right) + r_{2,k} \sin \left(x \sin \left(\frac{2\pi k}{m} \right) \right) \right].$$

It is known that the solution $S_m(x)$ of the form (11) for Problem 1 exists and is unique for $N+1 \geq m$ and the coefficients C_γ , r_1 , r_2 , r_{1k} and $r_{2k} \left(k = 1, 2, \dots, \frac{m}{2} - 1 \right)$ of the equation $S_m(x)$ are determined by the following system of $N+1+m$ linear equations:

$$\sum_{\gamma}^N C_\gamma G_m(x_\beta - x_\gamma) + Y_m(x_\beta) = \varphi(x_\beta), \quad \beta = \overline{0, N}, \quad (12)$$

$$\sum_{\gamma=0}^N C_{\gamma} e^{x_{\gamma}} = 0, \quad (13)$$

$$\sum_{\gamma=0}^N C_{\gamma} e^{-x_{\gamma}} = 0, \quad (14)$$

$$\sum_{\gamma=0}^N C_{\gamma} e^{x_{\gamma} \cos\left(\frac{2\pi k}{m}\right)} \cos\left(x \sin\left(\frac{2\pi k}{m}\right)\right) = 0, \quad k = 1, \overline{\frac{m}{2}-1}, \quad (15)$$

$$\sum_{\gamma=0}^N C_{\gamma} e^{x_{\gamma} \cos\left(\frac{2\pi k}{m}\right)} \sin\left(x \sin\left(\frac{2\pi k}{m}\right)\right) = 0, \quad k = 1, \overline{\frac{m}{2}-1}. \quad (16)$$

It is easy to show that the spline $S_m(x)$, defined by equation (11) with coefficients C_{γ} , r_1 , r_2 , r_{1k} and $r_{2k} \left(k = 1, 2, \dots, \frac{m}{2}-1 \right)$ satisfies conditions (i)–(v).

Conclusion. The investigation concludes with the derivation of explicit systems of linear algebraic equations that uniquely determine the coefficients for L-splines in $W_2^{(m,0)}$ spaces. The findings demonstrate that by utilizing the adjoint operator and fundamental solutions, it is possible to construct splines that are exact for power, trigonometric, and exponential functions – a significant improvement over standard polynomial interpolation. The existence and uniqueness of the solution $\delta(x)$ for any given partition on the interval $[0,1]$ confirm the stability of the proposed mathematical model. Ultimately, these results provide a robust computational tool for high-precision data modeling in fields such as signal processing, medical visualization, and numerical simulations, where capturing complex periodic patterns is critical.

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SPECTRAL PROPERTIES OF LINEAR OPERATORS IN HILBERT SPACES

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Abstract. This paper investigates the spectral properties of linear operators in Hilbert spaces, with particular emphasis on the distinction between finite- and infinite-dimensional cases. It introduces key concepts such as eigenvalues, spectrum, resolvent, and continuous spectrum, and explains their interrelations. Special attention is given to the classification of spectrum into point and continuous parts, highlighting the unique features arising in infinite-dimensional spaces. Several illustrative examples are provided to demonstrate the theoretical results and their applications.

Key words. Hilbert space, linear operator, eigenvalue, spectrum, regular point.

СПЕКТРАЛЬНЫЕ СВОЙСТВА ЛИНЕЙНЫХ ОПЕРАТОРОВ В ГИЛЬБЕРТОВЫХ ПРОСТРАНСТВАХ

Аннотация. В данной работе исследуются спектральные свойства линейных операторов в гильбертовых пространствах с особым акцентом на различие между конечномерными и бесконечномерными случаями. Вводятся основные понятия, такие как собственные значения, спектр, резольвента и непрерывный спектр, а также объясняется их взаимосвязь. Особое внимание уделяется классификации спектра на точечный и непрерывный, с выделением специфических особенностей, возникающих в бесконечномерных пространствах. Приводятся иллюстративные примеры для демонстрации теоретических результатов и их применений.

Ключевые слова: гильбертово пространство, линейный оператор, собственное значение, спектр, регулярная точка.

GILBERT FAZOLARIDA CHIZIQLI OPERATORLARNING SPEKTRAL XOSSALARI

Annotatsiya. Ushbu maqolada Gilbert fazolarida chiziqli operatorlarning spektral xossalari o'rganiladi, bunda ayniqsa chekli va cheksiz o'lchamli holatlar o'rtasidagi farqlarga alohida e'tibor qaratiladi. Xos qiymatlar, spektr, rezolventa va uzluksiz spektr kabi asosiy tushunchalar kiritilib, ularning o'zaro bog'liqligi tushuntiriladi. Spektrning nuqtaviy va uzluksiz qismlarga ajratilishi batafsil ko'rib chiqilib, cheksiz o'lchamli fazolarda yuzaga keladigan o'ziga xos xususiyatlar yoritiladi. Nazariy natijalarni va ularning qo'llanilishini ko'rsatish uchun misollar keltiriladi.

Kalit so'zlar: Gilbert fazosi, chiziqli operator, xos qiymat, spektr, regulyar nuqta.

In the theory of linear operators and their applications, a central role is played by the notion of the "spectrum" of an operator. In talking about the spectrum of an operator, it will always be tacitly assumed that the operator is defined on a complex space. Let A be a linear operator mapping a topological linear space E into itself. Then a number λ is called an eigenvalue of A if the equation

$$Ax = \lambda x$$

has at least one nonzero solution, and every such solution x is called an eigenvector of A (corresponding to the eigenvalue λ). Suppose E is finite-dimensional. Then the set of all eigenvalues of A is called the spectrum of A , and all other values of λ are said to be regular (points). In other words, λ is regular if and only if the operator $A - \lambda I$ is invertible, where I is identity operator. The operator $(A - \lambda I)^{-1}$ is then automatically bounded, like every operator on a finite-dimensional space.

Thus there are just two possibilities in the finite-dimensional case:

- 1) The equation $Ax = \lambda x$ has a nonzero solution, i.e., λ is an eigenvalue of A , so that the operator $(A - \lambda I)^{-1}$ fails to exist;
- 2) The operator $(A - \lambda I)^{-1}$ exists and is bounded, i.e., λ is a regular point.

However, in the case where E is infinite-dimensional, there is a third possibility:

3) The operator $(A - \lambda I)^{-1}$ exists (i.e., the equation $Ax = \lambda x$ has no nonzero solutions), but is not bounded.

To describe this more general situation, we introduce some new terminology and make an important modification in the definition of the spectrum.

Given an operator A mapping a (complex) topological linear space E into itself, the operator

$$R_\lambda = (A - \lambda I)^{-1} \quad (1)$$

is called the resolvent of A . The values of λ for which R_λ is defined for all E and continuous are said to be regular {points} of A , and the set of all other values of λ is called the spectrum of A . The eigenvalues of A still belong to the spectrum, since if $(A - \lambda I)x = 0$ for some $x \neq 0$, then (1) fails to exist. The set of all these eigenvalues is now called the point spectrum, and the rest of the spectrum is called the continuous spectrum. In other words, the continuous spectrum consists of all λ for which (1) exists but fails to be continuous. Thus there are now exactly three possibilities for any given value of λ :

- 1) λ is a regular point;
- 2) λ is an eigenvalue;
- 3) λ is a point of the continuous spectrum.

Denote by $\sigma_p(A)$, $\sigma_c(A)$, $\sigma(A)$ and $\rho(A)$ point spectrum, continuous spectrum, spectrum and resolvent set of A , respectively.

So, $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$, $\rho(A) = C/\sigma(A)$.

The possibility of an operator having a continuous spectrum is a characteristic feature of the theory of operators in infinite-dimensional spaces, distinguishing it from the finite-dimensional case.

In this thesis we will give spectral properties of linear operators in Hilbert spaces.

1. Let $H \cong C^n$ and let $A: H \rightarrow H$ be a linear operator (matrix). Then, in finite dimensions spectrum equals eigenvalues, that is $\sigma(A) = \{\lambda \in C: \det(A - \lambda I) = 0\}$.

2. The spectrum is always non-empty, $\sigma(A) \neq \emptyset$ and the number of elements of $\sigma(A)$ is not greater than n , that is $|\sigma(A)| \leq n$.

3. Algebraic and geometric multiplicity have the following relations for each eigenvalue λ :

$$1 \leq \dim \ker (A - \lambda I) \leq \text{Algebraic multiplicity}.$$

Note that algebraic multiplicity is multiplicity as a root of the characteristic polynomial and $\dim \ker (A - \lambda I)$ is geometric multiplicity.

4. A linear operator A on an n -dimensional space is diagonalizable if and only if:

$$\sum_{\lambda \in \sigma(A)} \dim \ker (A - \lambda I) = n,$$

equivalently:

$$H = \bigoplus_{\lambda \in \sigma(A)} E_\lambda,$$

where $E_\lambda = \ker (A - \lambda I)$.

5. Let H be a Hilbert space and let $A: H \rightarrow H$ be a linear operator (matrix). If A is self-adjoint operator,

then all eigenvalues of A are real, i.e. $\sigma(A) \subset R$.

6. Spectral radius of a linear operator A on a finite-dimensional Hilbert space is:

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

Example 1: Consider the linear operator represented by the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

1. **Characteristic polynomial:**

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

Since the matrix is triangular:

$$\det(A - \lambda I) = (2 - \lambda)^2(3 - \lambda)$$

2. **Spectrum (eigenvalues):**

$$(2 - \lambda)^2(3 - \lambda) = 0$$

Hence:

$$\sigma(A) = \{2, 3\}.$$

In this case, algebraic multiplicities are 2 and 1 for $\lambda = 2$ and $\lambda = 3$, respectively.

3. **Geometric Multiplicity**

As mentioned above, geometric multiplicity of A is $\dim \ker(A - \lambda I)$ for λ .

So, for $\lambda = 2$

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By solving the equation $(A - 2I)x = 0$, we will have $\ker(A - 2I) = \{x = (t, 0, 0) : t \in \mathbb{R}\}$ and $\dim \ker(A - \lambda I) = 1$. Consequently, geometric multiplicity of A is 1 for $\lambda = 2$.

For $\lambda = 3$

$$A - 3I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\ker(A - 3I) = \{x = (0, 0, t) : t \in \mathbb{R}\}$ and $\dim \ker(A - 3I) = 1$. Consequently, geometric multiplicity of A is 1 for $\lambda = 3$.

4. **Spectral radius is**

$$r(A) = \max\{|2|, |3|\} = 3.$$

Let us consider next examples for infinite case.

Example 2. Consider linear operator A on Hilbert space l_2 as:

$$A(x_1, x_2, x_3, \dots) = (2x_1, 3x_2, 4x_3, \dots).$$

By solving the equation $Ax = \lambda x$ we can get all eigenvalues as following: $\lambda = 2, 3, 4, \dots$ i.e. point spectrum consists of these numbers. We can show that continuous spectrum of this operator is empty. Indeed, for any $\lambda \in \mathbb{C} \setminus \{2, 3, 4, \dots\}$ the operator $A - \lambda I$ is invertible and R_λ is well defined on l_2 as well as bounded with

$$\begin{aligned} R_\lambda(y_1, y_2, y_3, \dots) &= (A - \lambda I)^{-1}(y_1, y_2, y_3, \dots) = \\ &= \left(\frac{1}{2-\lambda}y_1, \frac{1}{3-\lambda}y_2, \frac{1}{4-\lambda}y_3, \dots \right). \end{aligned}$$

So, set of regular points of A is $\mathbb{C} \setminus \{2, 3, 4, \dots\}$.

Example 3. Let us consider linear operator A on Hilbert space l_2 as:

$$A(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots).$$

By solving the equation $Ax = \lambda x$ we can get all eigenvalues as following: $\lambda = 1, \frac{1}{2}, \frac{1}{3}, \dots$ and these numbers belong to point spectrum, i.e. $\sigma_p(A) = \{\frac{1}{n} : n \in \mathbb{N}\}$. Continuous spectrum $\sigma_c(A)$ of A is $\sigma_c(A) = \{0\}$. Hence, the spectrum of A is

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) = \{\frac{1}{n} : n \in N\} \cup \{0\}.$$

Resolvent set $\rho(A)$ of A is $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

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ОБ ОДНОЙ ОБРАТНОЙ ЗАДАЧЕ ОПРЕДЕЛЕНИЯ ИСТОЧНИКА В ОДНОМЕРНОЙ СИСТЕМЕ ПОРОУПРУГОСТИ

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Аннотация. В данной работе рассматривается обратная задача определения распределённого источника в системе пороупругости, описываемой тремя упругими параметрами в бездиссипативном приближении. Доказана теорема о существовании и единственности решения рассматриваемой задачи в классе $C^{3,2}(\mathbb{R} \times [0, x_0])$.

Ключевые слова: пористая среда, задача Коши, прямая задача, обратная задача, медленная волна, пористость.

ON AN INVERSE SOURCE DETERMINATION PROBLEM IN A ONE-DIMENSIONAL POROELASTICITY SYSTEM

Abstract. This paper considers the inverse problem of determining a distributed source in a poroelasticity system described by three elastic parameters in the dissipationless approximation. A theorem on the existence and uniqueness of the solution to the considered problem in the class $C^{3,2}(\mathbb{R} \times [0, x_0])$ is proved.

Keywords: porous medium, Cauchy problem, direct problem, inverse problem, slow wave, porosity.

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Annotatsiya. Ushbu ishda dissipatsiyasiz yaqinlashuvda uchta elastik parametrlari bilan ifodalangan g'ovak-elastiklik sistemasida taqsimlangan manbani aniqlash teskari masalasi qaralgan. Qaralayotgan masalaning $C^{3,2}(\mathbb{R} \times [0, x_0])$ sinfida yechimining mavjudligi va yagonaligi haqidagi teorema isbotlangan.

Kalit so'zlar: g'ovak muhit, Koshi masalasi, to'g'ri masala, teskari masala, sekin to'lqin, g'ovaklilik.

Введение. В прикладных задачах динамики волн часто возникает необходимость учета пористости среды, ее флюидонасыщенности и гидродинамического фона. В частности, подобные вопросы возникают в разведочной геофизике при поиске нефтяных пластов, а также при выборе методов волнового воздействия на нефтяные и газовые месторождения с целью интенсификации добычи. Аналогичные задачи возникают и в сейсмологии при геофизическом мониторинге свойств очаговой зоны для прогноза землетрясений [1–2].

В геофизике динамические и кинематические характеристики упругих волн, распространяющихся в фрагментированных флюидонасыщенных горных породах, несут важную информацию о строении, составе и условиях залегания пород. Они также содержат сведения о литологии пород, характере их границ, трещиноватости, пористости, наличии различных нарушений и локальных включений, а также о составе и фазовом состоянии флюидов, заполняющих поровое пространство коллекторов. Математические модели в теории волн позволяют определять численные значения скоростей распространения и коэффициентов поглощения упругих сейсмических волн в зависимости от вещественного состава флюидонасыщенного коллектора, его внутреннего строения и воздействия окружающей среды. Чем точнее определяются значения скорости распространения и коэффициента поглощения упругих сейсмических волн, тем более реалистичной и адекватной является математическая модель.

Выявленные особенности поглощения сейсмических волн в трещиновато-пористых средах, а также одновременное проявление множественных электросейсмических эффектов не удается согласовать с простейшими моделями идеально упругой изотропной среды или среды Био. Реальные

геологические среды являются многофазными, электропроводящими, трещиноватыми, пористыми и обладают рядом других сложных свойств [4–8].

В работе [9] построена нелинейная математическая модель насыщенной жидкостью пористой упругодеформируемой среды. Данная модель основана на трех фундаментальных принципах: выполнении законов сохранения, принципе относительности Галилея, а также согласованности уравнений движения насыщающей жидкости с условиями термодинамического равновесия.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} &= 0, \quad \frac{\partial S}{\partial t} + \operatorname{div} \left(\frac{S}{\rho} \mathbf{j} \right) = 0, \quad \mathbf{j} = \rho_s \mathbf{u}_1 + \rho_l \mathbf{u}_2, \\ \frac{\partial \rho_l}{\partial t} + \operatorname{div}(\rho_l \mathbf{u}_2) &= 0, \quad \frac{\partial g_{ik}}{\partial t} + g_{kj} \partial_i u_{1j} + g_{ij} \partial_k u_{1j} + u_{1j} \partial_j g_{ik} = 0, \\ \frac{\partial e}{\partial t} + \operatorname{div} \mathbf{Q} &= 0, \quad \rho_s = \operatorname{const} \sqrt{\det(g_{ik})}, \\ \frac{\partial j_i}{\partial t} + \partial_k \left(\rho_s u_{1i} u_{1k} + \rho_s u_{2i} u_{2k} + p \delta_{ik} + h_{ij} g_{jk} \right) &= 0, \\ Q_k &= \left(\hat{\mu} + \frac{u_2^2}{2} + \frac{TS}{\rho} \right) j_k + \rho_s (\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) + u_{1i} h_{km} g_{mi}, \\ \frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2, \nabla) \mathbf{u}_2 &= -\frac{\nabla p}{\rho} + \frac{\rho_s}{2\rho} \nabla (\mathbf{u}_1 - \mathbf{u}_2)^2 - \frac{h_{ik}}{2\rho} \nabla g_{ik}. \end{aligned}$$

Здесь $\mathbf{u}_1$ – скорость движения упругой пористой среды; $\mathbf{u}_2$ – скорость насыщающей жидкости, $\rho = \rho_l + \rho_s$, ρ_s , ρ_l – плотность континуума, парциальная плотность пористого тела, парциальная плотность жидкости соответственно; g_{ik} – метрический тензор упругой деформации; h_{ik} – тензор напряжений; e , S – энергия и энтропия единицы объёма; $\hat{\mu}$ – химический потенциал; T – температура; p – давление, $\mathbf{j}_0$ – относительная импульс. При этом выполняется первое начало термодинамики для рассматриваемой системы

$$de_0 = TdS + \hat{\mu}d\rho + (\mathbf{u}_1 - \mathbf{u}_2, d\mathbf{j}_0) + \frac{1}{2} h_{ik} dg_{ik}.$$

Прямая задача. Рассмотрим задачу о распространении волн на полуограниченной прямой $x \geq 0$ для пористой среде в обратимом приближении, описываемый одномерной однородной системой уравнений [2], [3], [6], [12], [13]:

$$\frac{\mathbb{I}^2 u_1(t, x)}{\mathbb{I} t^2} - a_{11} \frac{\mathbb{I}^2 u_1(t, x)}{\mathbb{I} x^2} - a_{12} \frac{\mathbb{I}^2 u_2(t, x)}{\mathbb{I} x^2} = F_1(t, x), \quad t \in \mathbb{R}, \quad 0 < x \leq x_0, \quad (1.1)$$

$$\frac{\mathbb{I}^2 u_2(t, x)}{\mathbb{I} t^2} - a_{21} \frac{\mathbb{I}^2 u_1(t, x)}{\mathbb{I} x^2} - a_{22} \frac{\mathbb{I}^2 u_2(t, x)}{\mathbb{I} x^2} = F_2(t, x), \quad t \in \mathbb{R}, \quad 0 < x \leq x_0, \quad (1.2)$$

где $\mathbf{F} = (F_1, F_2)$ – вектор массовых сил,

$$\begin{aligned} a_{11} &= \frac{\lambda + 2\mu}{\rho_s} + \left(\rho \alpha_3 + \frac{\lambda + 2\mu/3}{\rho^2} \right) \rho_s - \frac{\lambda + 2\mu/3}{\rho}, \\ a_{12} &= \left(\rho^2 \alpha_3 + \frac{\lambda + 2\mu/3}{\rho^2} - \frac{\lambda + 2\mu/3}{\rho_s} \right) \frac{\rho_l}{\rho}, \\ a_{21} &= \left(\rho^2 \alpha_3 + \frac{\lambda + 2\mu/3}{\rho} \right) \frac{\rho_l}{\rho} - \frac{\lambda + 2\mu/3}{\rho}, \quad a_{22} = \left(\rho^2 \alpha_3 + \frac{\lambda + 2\mu/3}{\rho} \right) \frac{\rho_l}{\rho}, \end{aligned}$$

$r_s = r_s^f (1 - d_0)$ и $r_l = r_l^f d_0$ – парциальные плотности, r_s^f и r_l^f – физические плотности

упругого пористого тела и жидкости, соответственно, $\rho = \rho_l + \rho_s$, d_0 – пористость, α_3 , λ , μ – упругие параметры пористой среды [7], [11].

Рассмотрим задачу Коши для системы уравнений пороупругости (1.1) и (1.2) со следующими данными Коши [11]:

$$u_1|_{x=0} = j_1(t), \quad t \in \mathbb{R}, \quad (1.3)$$

$$\frac{\partial u_1}{\partial x}\bigg|_{x=0} = y_1(t), \quad t \in \mathbb{R}, \quad (1.4)$$

$$u_2|_{x=0} = j_2(t), \quad t \in \mathbb{R}, \quad (1.5)$$

$$\frac{\partial u_2}{\partial x}\bigg|_{x=0} = y_2(t), \quad t \in \mathbb{R}. \quad (1.6)$$

Здесь $j_1(t)$, $j_2(t)$, $y_1(t)$, $y_2(t)$, $F_1(t, x)$ и $F_2(t, x)$ – заданные функции.

Обратная задача. В данной параграфе исследуется вопрос одновременного определения решения $u_1(t, x)$, $u_2(t, x)$ из системы уравнений (1.1), (1.2) и функции $f_1(x)$, $f_2(x)$ ($F_1(t, x) = g(t)f_1(x)$, $F_2(t, x) = g(t)f_2(x)$), если постоянные α_3 , λ , μ ,

ρ_s^f , ρ_l^f , d_0 , и функция $g(t)$ — фиксированы, заданы условия (1.3)-(1.6) и

$$u_1|_{t=0} = v_1(x), \quad 0 \leq x \leq x_0, \quad (2.1)$$

$$u_2|_{t=0} = v_2(x), \quad 0 \leq x \leq x_0. \quad (2.2)$$

Другими словами, задача заключается в определении функции $u_1(t, x)$, $u_2(t, x)$ и объемных сил вида

$$F_1(t, x) = g(t)f_1(x), \quad F_2(t, x) = g(t)f_2(x),$$

по приведенной выше информации. Такие задачи принято называть обратными для дифференциальных уравнений [11], [12], [13].

В дальнейшем предполагаем, что все рассматриваемые функции и вектор-функции дважды непрерывно-дифференцируемы по переменной x и трижды непрерывно-дифференцируемы по переменной по t . Кроме того, считаем известную функцию $g(t) \neq 0$ для $t \in \mathbb{R}$. Также предположим, что выполнены условия согласования данных.

Методология исследования. С помощью дифференцирования по времени обе части (1.1) и (1.2), исключением искомых функции $f_1(x)$, $f_2(x)$, а также с помощью выделения

$$v_1 = \frac{\partial u_1}{\partial t}, \quad v_2 = \frac{\partial u_2}{\partial t}$$

получим интегро-дифференциальную систему

$$\begin{aligned} a_{11} \frac{\partial^2 v_1}{\partial x^2} + a_{12} \frac{\partial^2 v_2}{\partial x^2} &= \frac{\partial^2 v_1}{\partial t^2} - \frac{g'}{g} \frac{\partial v_1}{\partial t} + \\ &+ \frac{g'}{g} \frac{\partial^2}{\partial x^2} \int_0^t (a_{11}v_1(\tau, x) + a_{12}v_2(\tau, x))d\tau + \frac{g'}{g} a_{11}v_1''(x) + \frac{g'}{g} a_{12}v_2''(x), \end{aligned} \quad (2.3)$$

$$a_{21} \frac{\partial^2 v_1}{\partial x^2} + a_{22} \frac{\partial^2 v_2}{\partial x^2} = \frac{\partial^2 v_2}{\partial t^2} - \frac{g'}{g} \frac{\partial v_2}{\partial t} + \frac{g'}{g} \frac{\partial^2}{\partial x^2} \int_0^t (a_{21} v_1(\tau, x) + a_{22} v_2(\tau, x)) d\tau + \frac{g'}{g} a_{21} v_1''(x) + \frac{g'}{g} a_{22} v_2''(x), \quad (2.4)$$

Так как детерминант матрицы $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ отличен от нуля [7], [17]. Вводя новые функции $\tilde{v}_1$ и $\tilde{v}_2$ вместо v_1 и v_2 по формуле [11]

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}, \quad T = \begin{pmatrix} c_{l_1}^2 - a_{22} & c_{l_2}^2 - a_{22} \\ a_{21} & a_{21} \end{pmatrix},$$

где c_{l_1} и c_{l_2} - скорости быстрой и медленной продольной волны в пористой среде [17]:

$$c_{l_1}^2 = B_* \left(1 + \sqrt{1 - \frac{b_*}{B_*^2}} \right), \quad c_{l_2}^2 = B_* \left(1 - \sqrt{1 - \frac{b_*}{B_*^2}} \right),$$

$$B_* = \frac{\alpha_3 \rho^2}{2} + \frac{\lambda + 2\mu/3}{2\rho} \left(\frac{\lambda + 2\mu/3}{\lambda + \frac{2}{3}\mu} \frac{\rho}{\rho_s} - 1 \right),$$

$$b_* = (\lambda + 2\mu) \left(\frac{1}{\rho_s} - \frac{1}{\rho} \right) \left[\frac{\lambda + 2\mu/3}{\rho} \left(1 - \frac{\lambda + \frac{2}{3}\mu}{\lambda + 2\mu} \right) + \alpha_3 \rho^2 \right].$$

После простых преобразований, получим задачу Коши для интегро-дифференциальной системы уравнений струны

$$\begin{aligned} \frac{\partial^2 \tilde{v}_1}{\partial x^2} &= \frac{1}{c_{l_1}^2} \frac{\partial^2 \tilde{v}_1}{\partial t^2} - \frac{g'}{g} \left[\frac{1}{c_{l_1}^2} \frac{\partial \tilde{v}_1}{\partial t} - \frac{\partial^2}{\partial x^2} \int_0^t \tilde{v}_1(\tau, x) d\tau - h_1(x) \right], \quad t \in \mathbb{R}, \quad x \in (0, x_0], \\ (2.5) \quad \frac{\partial^2 \tilde{v}_2}{\partial x^2} &= \frac{1}{c_{l_2}^2} \frac{\partial^2 \tilde{v}_2}{\partial t^2} - \frac{g'}{g} \left[\frac{1}{c_{l_2}^2} \frac{\partial \tilde{v}_2}{\partial t} - \frac{\partial^2}{\partial x^2} \int_0^t \tilde{v}_2(\tau, x) d\tau - h_2(x) \right], \quad t \in \mathbb{R}, \quad x \in (0, x_0], \quad (2.6) \end{aligned}$$

$$\tilde{v}_1|_{x=0} = \frac{1}{c_{l_1}^2 - c_{l_2}^2} \left(\phi_1'(t) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \phi_2'(t) \right), \quad t \in \mathbb{R}, \quad (2.7)$$

$$\frac{\partial \tilde{v}_1}{\partial x} \Big|_{x=0} = \frac{1}{c_{l_1}^2 - c_{l_2}^2} \left(\psi_1'(t) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \psi_2'(t) \right), \quad t \in \mathbb{R}, \quad (2.8)$$

$$\tilde{v}_2|_{x=0} = \frac{1}{c_{l_2}^2 - c_{l_1}^2} \left(\phi_1'(t) + (a_{22} - c_{l_1}^2) \phi_2'(t) \right), \quad t \in \mathbb{R}, \quad (2.9)$$

$$\frac{\partial \tilde{v}_2}{\partial x} \Big|_{x=0} = \frac{1}{c_{l_2}^2 - c_{l_1}^2} \left(\psi_1'(t) + (a_{22} - c_{l_1}^2) \psi_2'(t) \right), \quad t \in \mathbb{R}, \quad (2.10)$$

здесь

$$h_1(x) = \frac{1}{c_{l_1}^2} \left(\left(\frac{a_{11}}{c_{l_1}^2 - c_{l_2}^2} + \frac{a_{22} - c_{l_2}^2}{c_{l_1}^2 - c_{l_2}^2} \right) v_1''(x) + \left(\frac{a_{12}}{c_{l_1}^2 - c_{l_2}^2} + \frac{(a_{22} - c_{l_2}^2)a_{22}}{a_{21}(c_{l_1}^2 - c_{l_2}^2)} \right) v_2''(x) \right),$$

$$h_2(x) = \frac{1}{c_{l_2}^2} \left(\left(\frac{a_{11}}{c_{l_2}^2 - c_{l_1}^2} + \frac{c_{l_1}^2 - a_{22}}{c_{l_1}^2 - c_{l_2}^2} \right) v_1''(x) + \left(\frac{a_{12}}{c_{l_2}^2 - c_{l_1}^2} + \frac{(a_{22} - c_{l_2}^2)a_{22}}{a_{21}(c_{l_2}^2 - c_{l_1}^2)} \right) v_2''(x) \right).$$

Решения задачи Коши (2.5)-(2.10) даются формулами типа Даламбера

$$\begin{aligned} \tilde{v}_1 = & \frac{1}{2(c_{l_1}^2 - c_{l_2}^2)} \left(\phi_1'(t + \frac{x}{c_{l_1}}) + \frac{a_{22} - c_{l_1}^2}{a_{21}} \phi_2'(t + \frac{x}{c_{l_1}}) + \phi_1'(t - \frac{x}{c_{l_1}}) + \frac{a_{22} - c_{l_1}^2}{a_{21}} \phi_2'(t - \frac{x}{c_{l_1}}) \right) + \\ & + \frac{c_{l_1}}{2(c_{l_1}^2 - c_{l_2}^2)} \left(\psi_1(t + \frac{x}{c_{l_1}}) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \psi_2(t + \frac{x}{c_{l_1}}) \right) - \\ & - \frac{c_{l_1}}{2(c_{l_1}^2 - c_{l_2}^2)} \left(\psi_1(t - \frac{x}{c_{l_1}}) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \psi_2(t - \frac{x}{c_{l_1}}) \right) + \end{aligned} \quad (2.11)$$

$$\begin{aligned} & + \frac{c_{l_1}}{2} \int_0^x \int_{t-(x-y)/c_{l_1}}^{t+(x-y)/c_{l_1}} \frac{g'(\tau)}{g(\tau)} \frac{\partial^2}{\partial y^2} \int_0^\tau \tilde{v}_1(s, y) ds d\tau dy - \frac{1}{2c_{l_1}} \int_0^x \int_{t-(x-y)/c_{l_1}}^{t+(x-y)/c_{l_1}} \frac{g'(\tau)}{g(\tau)} \frac{\partial \tilde{v}_1(\tau, y)}{\partial \tau} d\tau dy + \\ & + \frac{c_{l_1}}{2} \int_0^x \int_{t-(x-y)/c_{l_1}}^{t+(x-y)/c_{l_1}} \frac{g'(\tau)}{g(\tau)} h_1(y) d\tau dy, \end{aligned}$$

$$\begin{aligned} \tilde{v}_2 = & \frac{1}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\phi_1'(t + \frac{x}{c_{l_2}}) + (a_{22} - c_{l_2}^2) \phi_2'(t + \frac{x}{c_{l_2}}) + \phi_1'(t - \frac{x}{c_{l_2}}) + (a_{22} - c_{l_2}^2) \phi_2'(t - \frac{x}{c_{l_2}}) \right) + \\ & + \frac{c_{l_2}}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\psi_1(t + \frac{x}{c_{l_2}}) + (a_{22} - c_{l_1}^2) \psi_2(t + \frac{x}{c_{l_2}}) \right) - \\ & - \frac{c_{l_2}}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\psi_1(t - \frac{x}{c_{l_2}}) + (a_{22} - c_{l_1}^2) \psi_2(t - \frac{x}{c_{l_2}}) \right) + \end{aligned} \quad (2.12)$$

$$\begin{aligned} & + \frac{c_{l_2}}{2} \int_0^x \int_{t-(x-y)/c_{l_2}}^{t+(x-y)/c_{l_2}} \frac{g'(\tau)}{g(\tau)} \frac{\partial^2}{\partial y^2} \int_0^\tau \tilde{v}_2(s, y) ds d\tau dy - \frac{1}{2c_{l_2}} \int_0^x \int_{t-(x-y)/c_{l_2}}^{t+(x-y)/c_{l_2}} \frac{g'(\tau)}{g(\tau)} \frac{\partial \tilde{v}_2(\tau, y)}{\partial \tau} d\tau dy + \\ & + \frac{c_{l_2}}{2} \int_0^x \int_{t-(x-y)/c_{l_2}}^{t+(x-y)/c_{l_2}} \frac{g'(\tau)}{g(\tau)} h_2(y) d\tau dy. \end{aligned}$$

Используя формулу Лейбница о дифференцировании интеграла с переменным пределом, получаем следующие соотношения:

$$a) \int_0^x \int_{t-(x-y)/c_{l_1}}^{t+(x-y)/c_{l_1}} \frac{g'(\tau)}{g(\tau)} \frac{\partial^2}{\partial y^2} \int_0^\tau \tilde{v}_1(s, y) ds d\tau dy = \int_0^x \frac{\partial}{\partial y} \left[\int_{t-(x-y)/c_{l_1}}^{t+(x-y)/c_{l_1}} \frac{g'(\tau)}{g(\tau)} \int_0^\tau \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds d\tau \right] dy +$$

$$+ \frac{1}{c_l} \int_0^x \frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \int_0^{t+(x-y)/c_l} \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds dy +$$

$$+ \frac{1}{c_l} \int_0^x \frac{g'(t - (x-y)/c_l)}{g(t - (x-y)/c_l)} \int_0^{t-(x-y)/c_l} \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds dy.$$

$$a_1) \int_0^x \frac{\partial}{\partial y} \left[\int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \frac{g'(\tau)}{g(\tau)} \int_0^\tau \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds d\tau \right] dy = - \int_{t-x/c_l}^{t+x/c_l} \frac{g'(\tau)}{g(\tau)} \int_0^\tau \frac{\partial \tilde{v}_1}{\partial y}(s, 0) ds d\tau.$$

$$a_2) \frac{1}{c_l} \int_0^x \frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \int_0^{t+(x-y)/c_l} \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds dy =$$

$$= \frac{1}{c_l} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \int_0^{t+(x-y)/c_l} \tilde{v}_1(s, y) ds \right) dy +$$

$$+ \frac{1}{c_l^2} \int_0^x \frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \tilde{v}_1(t + (x-y)/c_l, y) dy -$$

$$- \frac{1}{c_l} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \right) \int_0^{t+(x-y)/c_l} \tilde{v}_1(s, y) dy =$$

$$= \frac{1}{c_l} \frac{g'(t)}{g(t)} \int_0^t \tilde{v}_1(s, x) ds - \frac{1}{c_l} \frac{g'(t + x/c_l)}{g(t + x/c_l)} \int_0^{t+x/c_l} \tilde{v}_1(s, 0) ds +$$

$$+ \frac{1}{c_l^2} \int_0^x \frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \tilde{v}_1(t + (x-y)/c_l, y) dy -$$

$$- \frac{1}{c_l} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t + (x-y)/c_l)}{g(t + (x-y)/c_l)} \right) \int_0^{t+(x-y)/c_l} \tilde{v}_1(s, y) ds dy.$$

$$a_3) \frac{1}{c_l} \int_0^x \frac{g'(t - (x-y)/c_l)}{g(t - (x-y)/c_l)} \int_0^{t-(x-y)/c_l} \frac{\partial \tilde{v}_1}{\partial y}(s, y) ds dy =$$

$$= \frac{1}{c_l} \frac{g'(t)}{g(t)} \int_0^t \tilde{v}_1(s, x) ds - \frac{1}{c_l} \frac{g'(t - x/c_l)}{g(t - x/c_l)} \int_0^{t-x/c_l} \tilde{v}_1(s, 0) ds -$$

$$- \frac{1}{c_l^2} \int_0^x \frac{g'(t - (x-y)/c_l)}{g(t - (x-y)/c_l)} \tilde{v}_1(t - (x-y)/c_l, y) dy -$$

$$- \frac{1}{c_l} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t - (x-y)/c_l)}{g(t - (x-y)/c_l)} \right) \int_0^{t-(x-y)/c_l} \tilde{v}_1(s, y) ds dy.$$

$$\begin{aligned}
 \text{b) } \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \frac{g'(\tau)}{g(\tau)} \frac{\partial \tilde{v}_1(s, y)}{\partial \tau} d\tau dy &= \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \left(\frac{g'(\tau)}{g(\tau)} \tilde{v}_1(\tau, y) \right)'_{\tau} d\tau dy - \\
 - \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \left(\frac{g'(\tau)}{g(\tau)} \right)' \tilde{v}_1(\tau, y) d\tau dy &= \int_0^x \left(\frac{g'(t+(x-y)/c_l)}{g(t+(x-y)/c_l)} \tilde{v}_1(t+(x-y)/c_l, y) - \right. \\
 - \left. \frac{g'(t-(x-y)/c_l)}{g(t-(x-y)/c_l)} \tilde{v}_1(t-(x-y)/c_l, y) \right) dy &- \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \left(\frac{g'(\tau)}{g(\tau)} \right)' \tilde{v}_1(s, y) d\tau dy.
 \end{aligned}$$

С учётом приведённых выше соотношений формулы (2.11) и (2.12) можно записать в следующем виде:

$$\begin{aligned}
 \tilde{v}_1 &= \frac{1}{2(c_l^2 - c_{l_2}^2)} \left(\phi_1'(t + \frac{x}{c_l}) + \frac{a_{22} - c_{l_1}^2}{a_{21}} \phi_2'(t + \frac{x}{c_l}) + \phi_1'(t - \frac{x}{c_l}) + \frac{a_{22} - c_{l_1}^2}{a_{21}} \phi_2'(t - \frac{x}{c_l}) \right) + \\
 &+ \frac{c_{l_1}}{2(c_l^2 - c_{l_2}^2)} \left(\psi_1(t + \frac{x}{c_l}) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \psi_2(t + \frac{x}{c_l}) \right) - \\
 &- \frac{c_{l_1}}{2(c_l^2 - c_{l_2}^2)} \left(\psi_1(t - \frac{x}{c_l}) + \frac{(a_{22} - c_{l_2}^2)}{a_{21}} \psi_2(t - \frac{x}{c_l}) \right) - \\
 &- \frac{c_{l_1}}{2} \int_{t-x/c_l}^{t+x/c_l} \frac{g'(\tau)}{g(\tau)} \int_0^{\tau} \frac{\partial \tilde{v}_1}{\partial y}(s, 0) ds d\tau + \frac{g'(t)}{g(t)} \int_0^t \tilde{v}_1(s, x) ds - \frac{1}{2} \frac{g'(t+x/c_l)}{g(t+x/c_l)} \int_0^{t+x/c_l} \tilde{v}_1(s, 0) ds - \\
 &- \frac{1}{2} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t+(x-y)/c_l)}{g(t+(x-y)/c_l)} \right) \int_0^{t+(x-y)/c_l} \tilde{v}_1(s, y) ds dy - \quad (2.12) \\
 &- \frac{1}{2} \frac{g'(t-x/c_l)}{g(t-x/c_l)} \int_0^{t-x/c_l} \tilde{v}_1(s, 0) ds - \frac{1}{2} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t-(x-y)/c_l)}{g(t-(x-y)/c_l)} \right) \int_0^{t-(x-y)/c_l} \tilde{v}_1(s, y) ds dy - \\
 &- \frac{1}{2c_{l_1}} \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \left(\frac{g'(\tau)}{g(\tau)} \right)' \tilde{v}_1(s, y) d\tau dy + \frac{c_{l_1}}{2} \int_0^x \int_{t-(x-y)/c_l}^{t+(x-y)/c_l} \frac{g'(\tau)}{g(\tau)} h_1(y) d\tau dy, \\
 \tilde{v}_2 &= \frac{1}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\phi_1'(t + \frac{x}{c_{l_2}}) + (a_{22} - c_{l_2}^2) \phi_2'(t + \frac{x}{c_{l_2}}) + \phi_1'(t - \frac{x}{c_{l_2}}) + (a_{22} - c_{l_2}^2) \phi_2'(t - \frac{x}{c_{l_2}}) \right) + \\
 &+ \frac{c_{l_2}}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\psi_1(t + \frac{x}{c_{l_2}}) + (a_{22} - c_{l_1}^2) \psi_2(t + \frac{x}{c_{l_2}}) \right) - \\
 &- \frac{c_{l_2}}{2(c_{l_2}^2 - c_{l_1}^2)} \left(\psi_1(t - \frac{x}{c_{l_2}}) + (a_{22} - c_{l_1}^2) \psi_2(t - \frac{x}{c_{l_2}}) \right) +
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{c_{l_2}}{2} \int_{t-x/c_{l_2}}^{t+x/c_{l_2}} \frac{g'(\tau)}{g(\tau)} \int_0^\tau \frac{\partial \tilde{v}_2}{\partial y}(s, 0) ds d\tau + \frac{g'(t)}{g(t)} \int_0^t \tilde{v}_2(s, x) ds - \frac{1}{2} \frac{g'(t+x/c_{l_2})}{g(t+x/c_{l_2})} \int_0^{t+x/c_{l_2}} \tilde{v}_2(s, 0) ds - \\
 & -\frac{1}{2} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t+(x-y)/c_{l_2})}{g(t+(x-y)/c_{l_2})} \right) \int_0^{t+(x-y)/c_{l_2}} \tilde{v}_2(s, y) ds dy - \quad (2.13) \\
 & -\frac{1}{2} \frac{g'(t-x/c_{l_2})}{g(t-x/c_{l_2})} \int_0^{t-x/c_{l_2}} \tilde{v}_2(s, 0) ds - \frac{1}{2} \int_0^x \frac{\partial}{\partial y} \left(\frac{g'(t-(x-y)/c_{l_2})}{g(t-(x-y)/c_{l_2})} \right) \int_0^{t-(x-y)/c_{l_2}} \tilde{v}_2(s, y) ds dy - \\
 & -\frac{1}{2c_{l_2}} \int_0^x \int_{t-(x-y)/c_{l_2}}^{t+(x-y)/c_{l_2}} \left(\frac{g'(\tau)}{g(\tau)} \right)' \tilde{v}_2(\tau, y) d\tau dy + \frac{c_{l_2}}{2} \int_0^x \int_{t-(x-y)/c_{l_2}}^{t+(x-y)/c_{l_2}} \frac{g'(\tau)}{g(\tau)} h_2(y) d\tau dy.
 \end{aligned}$$

Данная система уравнений представляет систему уравнений типа Вольтерра второго рода. Известно, что такая система уравнений имеет единственное решения в окрестности начала координат

[18]. А решения задачи Коши (2.3), (2.4) с данными $v_k|_{x=0} = \varphi'_k(t)$, $\left. \frac{\partial v_k}{\partial x} \right|_{x=0} = \psi'_k(t)$, $k=1, 2$,

определяются согласно формулам (2.5) - (2.10):

$$\begin{aligned}
 v_1 &= (c_{l_1}^2 - a_{22}) \tilde{v}_1 + (c_{l_2}^2 - a_{22}) \tilde{v}_2, \\
 v_2 &= a_{21} (\tilde{v}_1 + \tilde{v}_2).
 \end{aligned} \quad (2.13)$$

Основной результат. Таким образом, доказана следующая теорема.

Теорема. Пусть $j_i(t) \in OC^3(\mathbb{R})$, $y_i(t) \in OC^2(\mathbb{R})$, $(i=1, 2)$ и $g(t) \in OC^2(\mathbb{R})$, кроме того $g(t) \neq 0$. Пусть также $v_i(x) \in C^2([0, x_0])$, $(i=1, 2)$ и удовлетворяют условиям согласования $\varphi_i(0) = v_i(0)$, $(i=1, 2)$. Тогда обратная задача (1.1)-(1.6), (2.1), (2.2) имеет единственное решения в окрестности начала координат в классе $u_i(t, x) \in C^{3,2}(\mathbb{R} \times [0, x_0])$, $f_i(x) \in C[0, x_0]$, $i=1, 2$.

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О РАЗРЕШИМОСТИ СМЕШАННОГО ЗАДАЧИ ДЛЯ НЕЛИНЕЙНОГО УРАВНЕНИЯ $K\partial\Phi(+1) - K\partial\Phi(-1)$ С ИСТОЧНИКОМ

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Аннотация. Для интегрирования нелинейного уравнения $K\partial\Phi(+1) - K\partial\Phi(-1)$ с источником в виде ряда применён метод обратной спектральной задачи. Выведена бесконечная система дифференциальных уравнений Дубровина, описывающая эволюцию спектральных данных оператора Хилла. Доказана разрешимость задачи Коши для этой системы уравнений в классе $C^4(R)$ непрерывно дифференцируемых периодических бесконечнозонных функций. Найдено точное решение нелинейного уравнения $K\partial\Phi(+1) - K\partial\Phi(-1)$ с источником в случае наличия одной нетривиальной лакуны у оператора Хилла.

Ключевые слова: Обратная спектральная задача, уравнение Хилла, нелинейное уравнения Кортевега-де Фриза ($K\partial\Phi(+1)$), нелинейное уравнения Кортевега-де Фриза отрицательного порядка $K\partial\Phi(-1)$, формулы следов, спектральные данные.

ON THE SOLVABILITY OF A MIXED PROBLEM FOR A NONLINEAR $KDV(+1) - KDV(-1)$ EQUATION WITH A SOURCE

Abstract. The inverse spectral problem method is applied to integrate the nonlinear $KdF(+1) - KdF(-1)$ equation with a series source. An infinite Dubrovin system of differential equations is derived, describing the evolution of the spectral data of the Hill operator. The solvability of the Cauchy problem for this system of equations in the class of continuously differentiable periodic infinite-gap functions is proved. An exact solution to the nonlinear $KdV(+1) - KdV(-1)$ equation with a source is found in the case of the presence of one nontrivial gap in the Hill operator.

Keywords: Inverse spectral problem, Hill equation, nonlinear Korteweg-de Vries equation ($KdV(+1)$), negative order nonlinear Korteweg-de Vries equation $KdV(-1)$, trace formulas, spectral data.

MANBALI NOCHIZIQLI $KDV(+1) - KDV(-1)$ TENGLAMASI UCHUN ARALASH MASALANING YECHILISHI HAQIDA

Annotatsiya. Teskari spektral masala usuli chiziqli bo'lmagan manbali $KdF(+1) - KdF(-1)$ tenglamasi uchun aralash masala yechimga egaligi o'rganilgan. Hill operatorining spektral berilganlarining evolyutsiyasini tavsiflovchi cheksizta Dubrovin differensial tenglamalar sistemasi olindi. Uzluksiz differentsiallanuvchi davriy cheksiz zonali funksiyalar sinfidagi bu tenglamalar sistemasi uchun Koshi masalasining yagona global yechimga egaligi isbotlangan. Nochiziqli manbali $KdF(+1) - KdF(-1)$ tenglamasining aniq yechimi Hill operatori spektrida bitta notrivial lakunasi bo'lgan holda topilgan.

Kalit so'zlar: Teskari spektral masala, Hill tenglamasi, chiziqli bo'lmagan Korteweg-de Vries tenglamasi ($KdF(+1)$), manfiy tartibli chiziqli bo'lmagan Korteweg-de Vries tenglamasi $KdF(-1)$, iz formulalari, spektral berilganlar.

Введение. Известно (см. [1]), что теория рекурсивных операторов и иерархий интегрируемых нелинейных систем, играют значительную роль при интегрировании методом обратной спектральной задачи. Следует отметить, что рекурсивные операторы допускают не только положительные, но и отрицательные степени. Этот подход был применён в работе [2], где исследованы отрицательные степени рекурсивного оператора $K\partial\Phi$ – типа уравнений. Это позволило рассматривать так называемые отрицательные потоки интегрируемых иерархий, которые существенно расширяют класс интегрируемых эволюционных уравнений. Нелинейное уравнение Кортевега–де Фриза отрицательного порядка обозначается через $K\partial\Phi(-1)$:

$$q_t = (p^2)_x, \quad p_{xx} = pq, \quad q = q(x, t), \quad p = p(x, t).$$

Комбинируя фокусирующее нелинейное уравнение Шредингера (ФНУШ) и комплексное модифицированное уравнение Кортевега–де Фриза (кмКдФ), в 1973 г. Хирота [3] предложил метод нахождения N-солитонных решения так называемого уравнения Хироты (ФНУШ–кмКдФ).

$$iu_t + \alpha(u_{xx} + 2|u|^2 u) + i\beta(u_{xxx} + 6|u|^2 u_x) = 0, \quad \alpha, \beta \in R, \quad x \in R, \quad t > 0.$$

Опирая на идеи работы [3], авторы статей [4]–[6] предложили метод построения точных решений задачи Коши для уравнений типа Хироты (ДНУШ–кмКдФ) следующих видов соответственно

$$iu_t + b(t)[u_{xx} - 2|u|^2 u] - ia(t)[u_{xxx} - 6|u|^2 u_x] = 0, \quad i = \sqrt{-1},$$

$$iu_t + b(t)[u_{xx} - 2(\rho^2 - |u|^2)u] - ia(t)[u_{xxx} - 6(\rho^2 - |u|^2)u_x] = 0, \quad 0 \leq \rho < \infty,$$

$$iu_t + b(t)[u_{xx} - 2|u|^2 u] - ia(t)[u_{xxx} - 6|u|^2 u_x] - ic(t)u_x - d(t)u = 0$$

в классе π – периодических бесконечнозонных функций. После этого авторы работ [7]–[11] установили разрешимость смешанной задачи для следующих комбинированных уравнений мКдФ–cosh Γ , мКдФ–Л и омКдФ–cosh Γ , омКдФ–cosh Γ

$$u_{xt} = a(t)\left(u_{xxxx} - \frac{3}{2}u_x^2 u_{xx}\right) + b(t)\cosh u,$$

$$u_{xt} = a(t)\left(u_{xxxx} - \frac{3}{2}u_x^2 u_{xx}\right) + b(t)e^u,$$

$$u_{xt} = a(t)e^u, \quad u = u(x, t), \quad x \in R, \quad t > 0,$$

$$\begin{cases} u_{xt} - e^{2u} - a(t)[u_{xxx} - 2(u_x \mu_{xt})_x] = 0, \\ \mu_{xx} = u^2, \quad u = u(x, t), \quad \mu = \mu(x, t), \quad x \in R, \quad t > 0 \end{cases}$$

$$\begin{cases} u_{xt} - \cosh(2u) - b(t)[u_{xxx} - 2(u_x \mu_{xt})_x] = 0, \\ \mu_{xx} = u^2, \quad u = u(x, t), \quad \mu = \mu(x, t), \quad x \in R, \quad t > 0 \end{cases}$$

в классе действительных бесконечнозонных π – периодических по x функций. Здесь $a(t), b(t), c(t), d(t) \in C([0, \infty))$ – заданные непрерывные ограниченные функции.

Постановка задачи. В настоящей работе рассматривается смешанная задача для уравнения, составленного из комбинаций нелинейных уравнений КдФ(+1) и КдФ(–1) (КдФ(+1) –КдФ(–1)) с источником следующего вида

$$\begin{cases} q_t = (p^2)_x + q_{xxx} - 6qq_x + 2 \sum_{k=0}^{\infty} \alpha_k s(\pi, \lambda_k, t) \frac{\partial}{\partial x} (\psi_+^2(x, \lambda_k, t)), \\ p_{xx} = pq, \quad q = q(x, t), \quad p = p(x, t), \quad x \in R, \quad t > 0 \end{cases} \quad (1)$$

при условиях

$$\begin{aligned} q(x, t)|_{t=0} &= q_0(x), \quad q_0(x + \pi) = q_0(x) \in C^4(R), \\ q(x, t)|_{x=0} &= q_1(t), \quad q_1(t) \in C^1(t > 0) \cap C(t \geq 0), \\ p^2(x, t)|_{x=0} &= p_1(t), \quad p_1(t) \in C^1(t > 0) \cap C(t \geq 0), \\ |p_1(t)| &\leq M_1, \quad |q_1(t)| \leq M_2, \quad M_j > 0, \quad j = 1, 2 \end{aligned} \quad (2)$$

а также при условиях периодичности и гладкости

$$q(x + \pi, t) = q(x, t), \quad q(x, t) \in C_{x,t}^{3,1}(t > 0) \cap C(t \geq 0), \quad (3)$$

$$p^2(x + \pi, t) = p^2(x, t), p^2(x, t) \in C_{x,t}^{1,1}(t > 0) \cap C(t \geq 0).$$

В уравнении (1) $\{\alpha\}_{k=0}^{\infty}$ – заданная действительная последовательность чисел, удовлетворяющая условию

$$\alpha_k = O\left(\frac{1}{k^2}\right), k \rightarrow +\infty \quad (4)$$

при этом существует $k_0 \in N$ такое, что $\alpha_{k_0} \neq 0$.

Функции $\psi_{\pm}(x, \lambda, t)$ являются решениями Флоке уравнений Хилла

$$L(t)y \equiv -y'' + q(x, t)y = \lambda y, x \in R, t > 0 \quad (5)$$

и определяются следующим образом:

$$\psi_{\pm}(x, \lambda, t) = c(x, \lambda, t) + m^{\pm}(\lambda, t)s(x, \lambda, t). \quad (6)$$

Функция Вейля-Титчмарша определяются следующими формулами

$$m^{\pm}(\lambda, t) = \frac{s'(\pi, \lambda, t) - c(\pi, \lambda, t) \mp \sqrt{\Delta^2(\lambda) - 4}}{2s(\pi, \lambda, t)},$$

где функции $c(x, \lambda, t)$ и $s(x, \lambda, t)$ – решения уравнения (5) с начальными условиями

$$c(0, \lambda, \tau, t) = 1, c'(0, \lambda, \tau, t) = 0 \text{ и } s(0, \lambda, \tau, t) = 0, s'(0, \lambda, \tau, t) = 1$$

соответственно.

Функция

$$\Delta(\lambda, t) \equiv c(\pi, \lambda, t) + s'(\pi, \lambda, t)$$

называется функцией Ляпунова для уравнения Хилла (5), здесь, $\lambda_n = \lambda_n(t)$ – собственные значения периодической или антипериодической $(y(0, \lambda, t) = \pm y(\pi, \lambda, t), y'(0, \lambda, t) = \pm y'(\pi, \lambda, t))$ задачи для уравнения (5). Они совпадают с корнями уравнения $\Delta(\lambda, t) \mp 2 = 0$.

Поэтому решения Флоке $\psi_{\pm}(x, \lambda_k(t), t), k \geq 0$ удовлетворяют условиям $\psi_{\pm}(x + \pi, \lambda_k(t), t) = \psi_{\pm}(x, \lambda_k(t), t), k \geq 0$, следовательно, функции $\psi_{\pm}^2(x, \lambda_k, t), k \geq 0$ являются π – периодическими функциями.

Основным изучаемым объектом в этой статье является разрешимость смешанной задачи (1)–(4), то есть, установление минимальных требований гладкости для начальных данных, при которых существует единственное глобальное решение этой нелинейной задачи.

Следует отметить, что разрешимость смешанной задачи для линейных гиперболических и параболических уравнений изучена в работе [12].

В данной работе предлагается алгоритм построения периодических бесконечнозонных решений $((q(x, t), p^2(x, t)), \psi_{\pm}(x, \lambda_k, t), k \geq 0), x \in R, t > 0$, смешанной задачи (1)–(4) сведением её к обратной спектральной задаче для оператора Хилла:

$$L(\tau, t)y \equiv -y'' + q(x + \tau, t)y = \lambda y, x, \tau \in R, t > 0. \quad (7)$$

Обозначим через $\theta(x, \lambda, \tau, t)$ и $\varphi(x, \lambda, \tau, t)$ решения уравнения (7) с начальными условиями $\theta(0, \lambda, \tau, t) = 1, \theta'(x, \lambda, \tau, t) = 0$ и $\varphi(0, \lambda, \tau, t) = 0, \varphi'(0, \lambda, \tau, t) = 1$ соответственно.

Пусть $\lambda_n = \lambda_n(\tau, t)$, $n \geq 0$ – собственные значения периодической или антипериодической задачи $(y(0, \lambda, \tau, t) = \pm y(\pi, \lambda, \tau, t), y'(0, \lambda, \tau, t) = \pm y'(\pi, \lambda, \tau, t))$ для уравнения (7). Тогда они совпадают с корнями уравнения $\Delta(\lambda, \tau, t) \mp 2 = 0$, где $\Delta(\lambda, \tau, t) \equiv \theta(\pi, \lambda, \tau, t) + \varphi'(\pi, \lambda, \tau, t)$. Кроме того, выполняется равенства $\Delta(\lambda, \tau, t) = \Delta(\lambda, t)$. Поэтому $\lambda_n(\tau, t) = \lambda_n(t)$, $n \geq 0$. Интервалы $(-\infty, \lambda_0)$, $(\lambda_{2n-1}, \lambda_{2n})$, $n \in N$, называются лакунами.

Обозначим через $\xi_n(\tau, t)$, $n \in N$ собственные значения задачи Дирихле для уравнения (7) с граничными условиями $y(0, \lambda, \tau, t) = 0$, $y(\pi, \lambda, \tau, t) = 0$, а также имеем $\xi_n(\tau, t) \in [\lambda_{2n-1}, \lambda_{2n}]$, $n \geq 1$.

Определение 1. Числа $\xi_n(\tau, t)$, $n \in N$ и знаки $\sigma_n(\tau, t) = \text{sgn}\{\varphi'(\pi, \xi_n, \tau, t) - \theta(\pi, \xi_n, \tau, t)\} = \pm 1$, $n \geq 1$, называются спектральными параметрами оператора $L(\tau, t)$. Спектральные параметры и границы спектра $\lambda_n = \lambda_n(\tau, t)$, $n \geq 0$ называются спектральными данными оператора Хилла $L(\tau, t)$.

Определение 2. Задача нахождения спектральных данных оператора $L(\tau, t)$ называется прямой задачей, а задача восстановления потенциала $q(x + \tau, t)$ оператора $L(\tau, t)$ по спектральным данным $\{\lambda_{n-1}(\tau, t), \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in N\}$ обратной задачей.

Коэффициент $q(x + \tau, t)$ оператора $L(\tau, t)$ определяется однозначно по спектральным данным $\{\lambda_{n-1}(\tau, t), \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1\}$ (см. [13]).

Используя начальную функцию $q_0(x)$, $x \in R$, построим оператор Хилла $L(\tau, 0)$ в виде

$$L(\tau, 0)y \equiv -y'' + q_0(x + \tau)y = \lambda y, \quad x, \tau \in R.$$

Решив прямую задачу, находим спектральные данные $\{\lambda_{n-1}, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \geq 1\}$ оператора $L(\tau, 0)$. Кроме того, спектральные параметры $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1$ являются периодическими функциями: $\xi_n^0(\tau + \pi) = \xi_n^0(\tau)$, $\sigma_n^0(\tau + \pi) = \sigma_n^0(\tau)$, $\tau \in R$, $n \geq 1$.

3. Основной результат и обсуждение

Основной результат настоящей работы содержится в следующей теореме.

Теорема 1. Если начальные данные $q_0(x)$, $q_1(t)$, $p_1(t)$ и α_n , $n \geq 0$ удовлетворяют условиям

$$\begin{aligned} q_0(x + \pi) &= q_0(x) \in C^4(R), \quad p_1(t), q_1(t) \in C^1(t > 0) \cap C(t \geq 0), \\ |p_1(t)| &\leq M_1, \quad |q_1(t)| \leq M_2, \quad M_j > 0, \quad j = 1, 2, \\ \alpha_n &= O\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty. \end{aligned}$$

то существует однозначно определённые решения $(q(x, t), p^2(x, t), \psi_+(x, \lambda_n, t), n \geq 0)$, $x \in R$, $t > 0$ задачи (1) – (4), которые определяются, соответственно следующими формулами:

$$q(x, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(x, t)), \quad (8)$$

$$p^2(x, t) = \frac{2p_1(t) \xi_n(x, t)}{A_n(t) - q_1(t)} \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\xi_k(x, t)}{\xi_k(0, t)}. \quad (9)$$

Кроме того, функция $\psi_{\pm}(x, \lambda, t)$ определяется из уравнений Хилла

$$-\frac{d^2 y}{dx^2} + q(x, t)y = \lambda y, \quad q(x + \pi, t) = q(x, t), \quad x \in \mathbb{R}, \quad t > 0$$

по формуле (6).

Для доказательства этой теоремы нам потребуются вспомогательные леммы.

Сначала введем аналог бесконечную систему дифференциальных уравнений, описывающую эволюцию спектральных данных $\{\lambda_{n-1}(\tau, t), \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1\}$ оператора Хилла $L(\tau, t)$. Отметим, что система уравнений Дубровина для оператора Хилла с конечнозонным потенциалом решена в работах Б.А.Дубровина [14] в связи с интегрированием уравнения Кортевега – де Фриза (КдФ(+1)), а обратная спектральная задача для оператора Хилла изучена в работах [15] – [17].

Лемма 1. Пусть $(q(x, t), p^2(x, t), \psi_+(x, \lambda_n, t), n \geq 0)$, $x \in \mathbb{R}, t > 0$ – решение смешанной задачи (1) – (4). Тогда границы спектра $\lambda_n(\tau, t), n \geq 0$ оператора $L(\tau, t)$ не зависят от τ и t , т.е. $\lambda_n(\tau, t) = \lambda_n, n \geq 0$, а спектральные параметры $\xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1$, удовлетворяют двум системам дифференциальных уравнений Дубровина:

$$\frac{\partial \xi_n(\tau, t)}{\partial \tau} = 2(-1)^{n-1} \sigma_n(\tau, t) h_n(\xi), \quad n \geq 1, \quad (10)$$

$$\frac{\partial \xi_n(\tau, t)}{\partial t} = 2(-1)^n \sigma_n(\tau, t) h_n(\xi) g_n(\xi), \quad n \geq 1. \quad (11)$$

Кроме того, выполняются следующие начальные условия:

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), \quad n \geq 1, \quad (12)$$

при этом $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \geq 1$ – спектральные параметры оператора Хилла $L(\tau, 0)$.

Последовательности $h_n(\xi)$ и $g_n(\xi), n \geq 1$, определяются формулами

$$h_n(\xi) = \sqrt{(\xi_n(\tau, t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(\tau, t))} \cdot f_n(\xi),$$

$$f_n(\xi) = \sqrt{(\xi_n(\tau, t) - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau, t))(\lambda_{2k} - \xi_n(\tau, t))}{(\xi_k(\tau, t) - \xi_n(\tau, t))^2}}, \quad (13)$$

$$g_n(\xi) = \frac{1}{2\xi_n(\tau, t)} p^2(\tau, t) + 2q(\tau, t) + 4\xi_n(\tau, t) + \sum_{k=0}^{\infty} \frac{\alpha_k \varphi(\pi, \lambda_k, \tau, t)}{\xi_n(\tau, t) - \lambda_k}.$$

Используя формулы следов

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)), \quad (14)$$

$$q^2(\tau, t) - \frac{1}{2} q_{\tau\tau}(\tau, t) = \lambda_0^2 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau, t)), \quad (15)$$

$$p^2(\tau, t) = \frac{2p_1(t) \xi_n(\tau, t)}{A_n(t) - q_1(t)} \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\xi_k(\tau, t)}{\xi_k(0, t)}, \quad (16)$$

а также разложения

$$\varphi(\pi, \lambda, \tau, t) = \pi \prod_{k=1}^{\infty} \frac{\xi_k(\tau, t) - \lambda}{k^2}, \quad (17)$$

система уравнения Дубровина

$$\begin{aligned} \frac{\partial \xi_n(\tau, t)}{\partial t} = 2(-1)^n \sigma_n(\tau, t) h_n(\xi) & \left(\frac{1}{2\xi_n(\tau, t)} p^2(\tau, t) + 2q(\tau, t) + \right. \\ & \left. + 4\xi_n(\tau, t) + \sum_{k=0}^{\infty} \frac{\alpha_k \varphi(\pi, \lambda_k, \tau, t)}{\xi_k(\tau, t) - \lambda_k} \right), \quad n \geq 1 \end{aligned}$$

можно записать в виде

$$\frac{\partial \xi_n(\tau, t)}{\partial t} = 2(-1)^n \sigma_n(\tau, t) \sqrt{(\xi_n(\tau, t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(\tau, t))} \cdot f_n(\xi) g_n(\xi), \quad n \geq 1. \quad (18)$$

Кроме того, имеет место начальные условия

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), \quad n \geq 1. \quad (19)$$

Здесь

$$\begin{aligned} g_n(\xi) = \frac{p_1(t)}{A_n(t) - q_1(t)} \prod_{\substack{k=1, \\ k \neq n}}^{\infty} \frac{\xi_k(\tau, t)}{\xi_k(0, t)} + 4\xi_n(\tau, t) + 2\lambda_0 + 2 \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)) + \\ + 4\xi_n(\tau, t) + \frac{\pi}{n^2} \sum_{k=0}^{\infty} \alpha_k \left(\prod_{\substack{m=1, \\ m \neq n}}^{\infty} \frac{\xi_m(\tau, t) - \lambda_k}{m^2} \right), \quad n \geq 1. \end{aligned} \quad (20)$$

Будем считать, что все лакуны открыты, то есть $\lambda_{2n-1} < \lambda_{2n}$, $n \geq 1$. В этом бесконечнозонном случае, для доказательства существования и единственности решения задачи Коши (18), (19) воспользуемся замены переменных

$$\xi_n(\tau, t) = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1}) \sin^2 x_n(\tau, t), \quad n \geq 1. \quad (21)$$

Тогда задачу (18), (19) можно переписать в виде

$$\frac{dx(\tau, t)}{dt} = H(x(\tau, t)), \quad x(\tau, t)|_{t=0} = x^0(\tau), \quad (22)$$

где

$$\begin{aligned} H(x) &= (H_1(x), H_2(x), \dots, H_n(x), \dots), \\ H_n(x) &= (-1)^n \sigma_n^0(\tau) \overline{f_n}(x) \overline{g_n}(x), \quad n \geq 1, \\ \overline{f_n}(x) &= f_n(\lambda_1 + (\lambda_2 - \lambda_1) \sin^2 x_1(\tau, t), \dots), \quad n \geq 1, \\ \overline{g_n}(x) &= g_n(\lambda_1 + (\lambda_2 - \lambda_1) \sin^2 x_1(\tau, t), \dots), \quad n \geq 1, \end{aligned}$$

$$\begin{aligned}x &= x(\tau, t) = (x_1(\tau, t), x_2(\tau, t), \dots, x_n(\tau, t), \dots), \\x^0(\tau) &= (x_1^0(\tau), x_2^0(\tau), \dots, x_n^0(\tau), \dots), \\x_n^0(\tau) &= \arcsin \sqrt{\frac{\xi_n^0(\tau) - \lambda_{2n-1}}{\lambda_{2n} - \lambda_{2n-1}}}, \quad n \geq 1.\end{aligned}$$

Задача Коши (22) рассматривается в следующем банаховом пространстве

$$K = \left\{ x(\tau, t) = (x_1(\tau, t), x_2(\tau, t), \dots, x_n(\tau, t), \dots), \quad \|x\| = \sum_{n=1}^{\infty} (\lambda_{2n} - \lambda_{2n-1}) |x_n| < \infty \right\}.$$

Известно (см. [17]), что если начальная функция $q_0(x)$ удовлетворяет условию

$$q_0(x + \pi) = q_0(x) \in C^4(\square),$$

то для собственных значений периодической или антипериодической задач для уравнения Хилла

$$L(0, 0)y \equiv -y'' + q_0(x)y = \lambda y,$$

справедливы соотношения

$$\sqrt{\lambda_{2k-1}}, \sqrt{\lambda_{2k}} = k + \sum_{1 \leq 2j+1 \leq 6} a_{2j+1}(2k)^{-(2j+1)} \mp \frac{|l_4(2k)|}{(2k)^5} + \frac{\varepsilon_k^{\pm}}{k^6}, \quad (23)$$

$$\gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \frac{|l_4(2k)|}{2^5 k^4} + \frac{\delta_k}{k^5}, \quad (24)$$

где

$$\begin{aligned}l_4(2k) &= \frac{1}{\pi} \int_0^{\pi} q_0^{(4)}(x) e^{-i2kx} dx, \quad \sum_{k=0}^{\infty} |l_4(2k)|^2 < \infty, \\ \sum_{k=0}^{\infty} \delta_k^2 &< \infty, \quad \delta_k = \varepsilon_k^+ + \varepsilon_k^-.\end{aligned}$$

Отсюда, учитывая $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$, $n \geq 1$, получим

$$\inf_{k \neq n} |\xi_n(\tau, t) - \xi_k(\tau, t)| \geq a > 0. \quad (25)$$

Теперь с учётом (24) и (25) нетрудно доказать следующую лемму.

Лемма 2. Справедливы оценки

$$c_1 n \leq |\bar{f}_n(x)| \leq c_2 n, \quad \left| \frac{\partial \bar{f}_n(x)}{\partial x_m} \right| \leq c_3 \gamma_m \cdot n, \quad (26)$$

$$|\bar{g}_n(x)| \leq c_4 n^2, \quad \left| \frac{\partial \bar{g}_n(x)}{\partial x_m} \right| \leq c_5 \gamma_m, \quad (27)$$

где константы $c_j > 0$, $j = \overline{1, 5}$, не зависят от параметров m и n .

Лемма 3. Если начальные данные $q_0(t)$, $q_1(t)$, $p_1(t)$ и α_n , $n \geq 0$ удовлетворяют условиям

$$q_0(x + \pi) = q_0(x) \in C^4(\square), \quad p_1(t), \quad q_1(t) \in C^1(t > 0) \cap C(t \geq 0),$$

$$|p_1(t)| \leq M_1, \quad |q_1(t)| \leq M_2, \quad M_j > 0, \quad j = 1, 2,$$

$$\alpha_n = O\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty,$$

то вектор – функция $H(x)$ удовлетворяет условию Липшица в банаховом пространстве K , т.е. существует постоянная $l = \text{const} > 0$ такая, что для произвольных элементов $x, y \in K$ выполняется следующее неравенство:

$$\|H(x) - H(y)\| \leq l \|x - y\|,$$

где

$$l = A \sum_{n=1}^{\infty} n^3 \gamma_n. \quad (28)$$

Доказательство. Сначала, используя лемму 2, оценим производную по $x_m(\tau, t)$ функции $F_n(x) = \bar{f}_n(x) \cdot \bar{g}_n(x)$:

$$\left| \frac{\partial F_n(x)}{\partial x_m} \right| \leq \left| \frac{\partial \bar{f}_n(x)}{\partial x_m} \right| \cdot |\bar{g}_n(x)| + \left| \frac{\partial \bar{g}_n(x)}{\partial x_m} \right| \cdot |\bar{f}_n(x)| \leq c_3 c_4 n^3 \gamma_m + c_2 c_5 n \gamma_m \leq$$

$$\leq c_3 c_4 \gamma_m n^3 \left(1 + \frac{c_2 c_5}{c_3 c_4 n^2} \right) \leq c \gamma_m n^3,$$

где $c = \text{const} > 0$ не зависят от параметров m и n .

Далее, используя выражение

$$H(x(\tau, t)) = (-1)^n \sigma_n^0(\tau) F_n(x(\tau, t)), \quad n \geq 1$$

получим равенство

$$|H_n(x(\tau, t)) - H_n(y(\tau, t))| = |F_n(x(\tau, t)) - F_n(y(\tau, t))|.$$

Теперь, применим теорему Лагранжа о конечном приращении к функции $\varphi(t) = F_n(y + t(x - y))$ на отрезке $t \in [0, 1]$. Тогда получим равенство

$$\varphi(1) - \varphi(0) = \varphi'(t^*), \quad t^* \in (0, 1)$$

то есть

$$F_n(x) - F_n(y) = \sum_{m=1}^{\infty} \frac{\partial F_n(\theta)}{\partial x_m} (x_m - y_m),$$

где $\theta = y + t^*(x - y)$. Отсюда следует, что

$$|H_n(x) - H_n(y)| = |F_n(x) - F_n(y)| \leq \sum_{m=1}^{\infty} \left| \frac{\partial F_n(\theta)}{\partial x_m} \right| \cdot |x_m - y_m| \leq$$

$$\leq c n^3 \sum_{m=1}^{\infty} \gamma_m |x_m - y_m| = c n^3 \|x - y\|.$$

Теперь, оценим норму:

$$\begin{aligned} \|H(x) - H(y)\| &= \sum_{n=1}^{\infty} \gamma_n \|H_n(x) - H_n(y)\| \leq \\ &\leq \sum_{m=1}^{\infty} c n^3 \gamma_m \|x - y\| = l \|x - y\|. \end{aligned}$$

Теперь оценим норму:

$$l = c \sum_{n=1}^{\infty} n^3 \gamma_n < \infty.$$

С учётом оценок (24) доказывается сходимость ряда (28). Таким образом, условие Липшица выполняется. Поэтому решение задачи Коши (22) (т.е (18)-(19)) для всех $t > 0$ и $\tau \in \square$ существует и единственно. Лемма 3 доказана.

Согласно лемме 3 существует единственное глобальное решение задачи Коши (18) – (19), то есть существуют однозначно определённые спектральные данные

$$\{\lambda_{n-1}, \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1\} \text{ оператора Хилла } L(\tau, t).$$

Хорошо известно (см. [13]) что коэффициент $q(\tau, t)$ оператора Хилла $L(\tau, t)$ определяется по формуле (14) однозначно по спектральным данным $\{\lambda_{n-1}, \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1\}$. Далее восстановим функцию $p^2(\tau, t)$ по формуле (16). Затем находятся решения Флоке $\psi_{\pm}(\tau, \lambda, t)$, $\tau \in R, t > 0$ уравнений Хилла

$$L(t)y \equiv -\frac{d^2 y}{d\tau^2} + q(\tau, t)y = \lambda y, \tau \in R, t > 0,$$

где

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)).$$

Лемма 4. Функция $(q(\tau, t), p^2(\tau, t), \psi_{\pm}(\tau, \lambda, t), n \geq 0)$, $\tau \in R, t > 0$ построенная с помощью системы уравнений Дубровина (18), (19) и формулы следов (14), (16) удовлетворяют системы нелинейных уравнений (1).

Таким образом, теорема 1 и лемма 3 дают алгоритм нахождения периодических бесконечнозонных решения смешанной задачи (1)-(4).

1. Сначала находим спектральные данные $\{\lambda_{n-1}, \xi_n^0(\tau, t), \sigma_n^0(\tau, t) = \pm 1, n \geq 1\}$, оператора Хилла $L(\tau, 0)$.

2. Обозначим спектральные данные оператора Хилла $L(\tau, t)$ через $\{\lambda_{n-1}, \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1\}$. Теперь, решая задачу Коши (18), (19) для произвольного $\tau \in \square$, находим $\xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \geq 1$.

3. Из формул (14) и (16) определяем $q(\tau, t)$ и $p^2(\tau, t)$.

4. Затем, используя найденные функция $q(\tau, t)$, получим решение Флоке $\psi_{\pm}(\tau, t, \lambda)$ уравнений Хилла

$$-\frac{d^2 y}{d\tau^2} + q(\tau, t)y = \lambda y, \quad \tau \in \square, \quad t > 0.$$

4. Однозонное решение нелинейного уравнения КдФ(+1) – КдФ(-1) с источником

Пример 1. Рассмотрим однозонный потенциал $q_0(x)$ оператора Хилла $L(\tau, 0) \equiv -y'' + q_0(x + \tau)y$, $x \in \square$, $\tau \in \square$, задаваемый спектром $E = \square \setminus \{(-\infty, \lambda_0) \cup (\lambda_1, \lambda_2)\}$ и спектральными параметрами $\xi_1^0 \in [\lambda_1, \lambda_2]$ и $\sigma_1^0(\tau) = \pm 1$.

В этом случае система уравнений Дубровина (10) состоит из одного уравнения

$$\begin{cases} \frac{\partial \xi_1^0(\tau)}{\partial \tau} = -2\sigma_1^0(\tau) \sqrt{(\xi_1^0(\tau) - \lambda_0)(\xi_1^0(\tau) - \lambda_1)(\lambda_2 - \xi_1^0(\tau))} \\ \xi_1^0(\tau)|_{\tau=0} = \xi_1^0, \quad \sigma_1^0(\tau)|_{\tau=0} = \sigma_1^0, \end{cases} \quad (29)$$

где $\{\xi_1^0, \sigma_1^0 = \pm 1\}$ – спектральные параметры оператора $L(0, 0) \equiv -y'' + q_0(x)y$, а

$\{\xi_1(\tau), \sigma_1(\tau) = \pm 1\}$ – спектральные параметры оператора $L(\tau, 0)$.

Решая уравнения (29) имеем

$$\xi_1^0(\tau) = \frac{\lambda_0 + \lambda_1 + \lambda_2}{3} - \wp(-\tau + \tau_0)$$

Из формул следов

$$q_0(\tau) = \lambda_0 + \lambda_1 + \lambda_2 - 2\xi_1^0(\tau)$$

получаем

$$q_0(\tau) = 2\wp(-\tau + \tau_0) + \frac{\lambda_0 + \lambda_1 + \lambda_2}{3} \quad (30)$$

однозонный потенциал оператора Хилла $L(\tau, 0)$, здесь $\wp(z)$ – эллиптическая функция Вейерштрасса (см. [18]).

Пример 2. Рассмотрим следующую смешанную задачу

$$\begin{cases} q_t = (p^2)_x + q_{xxx} - 6qq_x + 2\alpha_0 s(\pi, \lambda_0, t)(\psi_+^2(x, \lambda_0, t))_x, \\ p_{xx} = pq, \quad q = q(x, t), \quad p = p(x, t), \quad x \in \square, \quad t > 0, \end{cases} \quad (31)$$

при условиях:

$$\begin{aligned} q(x, t)|_{t=0} &= q_0(x) \equiv \frac{\lambda_0 + \lambda_1 + \lambda_2}{3} + 2\wp(-x + x_0), \\ p^2(x, t)|_{x=0} &= p_1(t), \quad q(x, t)|_{x=0} = q_1(t), \\ p_1(t), q_1(t) &\in C^1(t > 0) \cap C(t \geq 0), \quad |p_1(t)| \leq M_1, \quad |q_1(t)| \leq M_2. \end{aligned} \quad (32)$$

Здесь $q_1(t)$ и $p_1(t)$ – заданные действительные ограниченные функции. В однозонном случае система уравнений Дубровина (18), (19), состоит из одного уравнения следующего вида

$$\begin{cases} \frac{\partial \xi_1(\tau, t)}{\partial t} = -2\sigma_1(\tau, t) \sqrt{(\xi_1(\tau, t) - \lambda_0)(\xi_1(\tau, t) - \lambda_1)(\lambda_2 - \xi_1(\tau, t))} \cdot \beta(t) \\ \xi_1(\tau, t)|_{t=0} = \xi_1^0(\tau), \quad \sigma_1(\tau, t)|_{t=0} = \sigma_1^0(\tau) \end{cases} \quad (33)$$

где

$$\beta(t) = \frac{2p_1(t)}{A_1 - q_1(t)} + 2(\lambda_0 + \lambda_1 + \lambda_2) + \pi \alpha_0 \prod_{j=2}^{\infty} \frac{\lambda_{2j-1} - \lambda_0}{j^2}$$

$$A_n = \lambda_0 + \lambda_{2n-1} + \lambda_{2n} + \sum_{\substack{k=1 \\ k \neq n}}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(0, t)),$$

$$A_1 = \lambda_0 + \lambda_1 + \lambda_2.$$

Решив уравнения (33), находим

$$\xi_1(\tau, t) = \frac{\lambda_0 + \lambda_1 + \lambda_2}{3} - \wp \left(2 \int_0^t \beta(s) ds - \tau + \tau_0 \right) \quad (34)$$

Подставив (34) в формулы следов

$$q(\tau, t) = \lambda_0 + \lambda_1 + \lambda_2 - 2\xi_1(\tau, t),$$

$$p^2(\tau, t) = \frac{2p_1(t)}{A_1 - q_1(t)} \cdot \xi_1(\tau, t),$$

получим решение смешанной задачи (31)–(32):

$$q(\tau, t) = \frac{\lambda_0 + \lambda_1 + \lambda_2}{3} + 2\wp \left(2 \int_0^t \beta(s) ds - \tau + \tau_0 \right)$$

$$p^2(\tau, t) = \frac{2p_1(t)}{A_1 - q_1(t)} \left(\frac{\lambda_0 + \lambda_1 + \lambda_2}{3} - \wp \left(2 \int_0^t \beta(s) ds - \tau + \tau_0 \right) \right)$$

Используя найденную функцию $q(\tau, t)$, получаем решение Флоке $\psi_+(\tau, \lambda, t)$ уравнений Хилла

$$-\frac{d^2 y}{d\tau^2} + q(\tau, t)y = \lambda y, \tau \in \mathbb{R}, \quad t > 0.$$

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SPECTRAL PROPERTIES OF LINEAR OPERATORS IN HILBERT SPACES

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Abstract. This paper investigates the spectral properties of linear operators in Hilbert spaces, with particular emphasis on the distinction between finite- and infinite-dimensional cases. It introduces key concepts such as eigenvalues, spectrum, resolvent, and continuous spectrum, and explains their interrelations. Special attention is given to the classification of spectrum into point and continuous parts, highlighting the unique features arising in infinite-dimensional spaces. Several illustrative examples are provided to demonstrate the theoretical results and their applications.

Key words. Hilbert space, linear operator, eigenvalue, spectrum, regular point.

СПЕКТРАЛЬНЫЕ СВОЙСТВА ЛИНЕЙНЫХ ОПЕРАТОРОВ В ГИЛЬБЕРТОВЫХ ПРОСТРАНСТВАХ

Аннотация. В данной работе исследуются спектральные свойства линейных операторов в гильбертовых пространствах с особым акцентом на различие между конечномерными и бесконечномерными случаями. Вводятся основные понятия, такие как собственные значения, спектр, резольвента и непрерывный спектр, а также объясняется их взаимосвязь. Особое внимание уделяется классификации спектра на точечный и непрерывный, с выделением специфических особенностей, возникающих в бесконечномерных пространствах. Приводятся иллюстративные примеры для демонстрации теоретических результатов и их применений.

Ключевые слова: гильбертово пространство, линейный оператор, собственное значение, спектр, регулярная точка.

GILBERT FAZOLARIDA CHIZIQLI OPERATORLARNING SPEKTRAL XOSSALARI

Annotatsiya. Ushbu maqolada Gilbert fazolarida chiziqli operatorlarning spektral xossalari o'rganiladi, bunda ayniqsa chekli va cheksiz o'lchamli holatlar o'rtasidagi farqlarga alohida e'tibor qaratiladi. Xos qiymatlar, spektr, rezolventa va uzluksiz spektr kabi asosiy tushunchalar kiritilib, ularning o'zaro bog'liqligi tushuntiriladi. Spektrning nuqtaviy va uzluksiz qismlarga ajratilishi batafsil ko'rib chiqilib, cheksiz o'lchamli fazolarda yuzaga keladigan o'ziga xos xususiyatlar yoritiladi. Nazariy natijalarni va ularning qo'llanilishini ko'rsatish uchun misollar keltiriladi.

Kalit so'zlar: Gilbert fazosi, chiziqli operator, xos qiymat, spektr, regulyar nuqta.

In the theory of linear operators and their applications, a central role is played by the notion of the "spectrum" of an operator. In talking about the spectrum of an operator, it will always be tacitly assumed that the operator is defined on a complex space. Let A be a linear operator mapping a topological linear space E into itself. Then a number λ is called an eigenvalue of A if the equation

$$Ax = \lambda x$$

has at least one nonzero solution, and every such solution x is called an eigenvector of A (corresponding to the eigenvalue λ). Suppose E is finite-dimensional. Then the set of all eigenvalues of A is called the spectrum of A , and all other values of λ are said to be regular (points). In other words, λ is regular if and only if the operator $A - \lambda I$ is invertible, where I is identity operator. The operator $(A - \lambda I)^{-1}$ is then automatically bounded, like every operator on a finite-dimensional space.

Thus there are just two possibilities in the finite-dimensional case:

- 1) The equation $Ax = \lambda x$ has a nonzero solution, i.e., λ is an eigenvalue of A , so that the operator $(A - \lambda I)^{-1}$ fails to exist;
- 2) The operator $(A - \lambda I)^{-1}$ exists and is bounded, i.e., λ is a regular point.

However, in the case where E is infinite-dimensional, there is a third possibility:

3) The operator $(A - \lambda I)^{-1}$ exists (i.e., the equation $Ax = \lambda x$ has no nonzero solutions), but is not bounded.

To describe this more general situation, we introduce some new terminology and make an important modification in the definition of the spectrum.

Given an operator A mapping a (complex) topological linear space E into itself, the operator

$$R_\lambda = (A - \lambda I)^{-1} \quad (1)$$

is called the resolvent of A . The values of λ for which R_λ is defined for all E and continuous are said to be regular {points} of A , and the set of all other values of λ is called the spectrum of A . The eigenvalues of A still belong to the spectrum, since if $(A - \lambda I)x = 0$ for some $x \neq 0$, then (1) fails to exist. The set of all these eigenvalues is now called the point spectrum, and the rest of the spectrum is called the continuous spectrum. In other words, the continuous spectrum consists of all λ for which (1) exists but fails to be continuous. Thus there are now exactly three possibilities for any given value of λ :

- 1) λ is a regular point;
- 2) λ is an eigenvalue;
- 3) λ is a point of the continuous spectrum.

Denote by $\sigma_p(A)$, $\sigma_c(A)$, $\sigma(A)$ and $\rho(A)$ point spectrum, continuous spectrum, spectrum and resolvent set of A , respectively.

So, $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$, $\rho(A) = C/\sigma(A)$.

The possibility of an operator having a continuous spectrum is a characteristic feature of the theory of operators in infinite-dimensional spaces, distinguishing it from the finite-dimensional case.

In this thesis we will give spectral properties of linear operators in Hilbert spaces.

1. Let $H \cong C^n$ and let $A: H \rightarrow H$ be a linear operator (matrix). Then, in finite dimensions spectrum equals eigenvalues, that is $\sigma(A) = \{\lambda \in C: \det(A - \lambda I) = 0\}$.

2. The spectrum is always non-empty, $\sigma(A) \neq \emptyset$ and the number of elements of $\sigma(A)$ is not greater than n , that is $|\sigma(A)| \leq n$.

3. Algebraic and geometric multiplicity have the following relations for each eigenvalue λ :

$$1 \leq \dim \ker (A - \lambda I) \leq \text{Algebraic multiplicity}.$$

Note that algebraic multiplicity is multiplicity as a root of the characteristic polynomial and $\dim \ker (A - \lambda I)$ is geometric multiplicity.

4. A linear operator A on an n -dimensional space is diagonalizable if and only if:

$$\sum_{\lambda \in \sigma(A)} \dim \ker (A - \lambda I) = n,$$

equivalently:

$$H = \bigoplus_{\lambda \in \sigma(A)} E_\lambda,$$

where $E_\lambda = \ker (A - \lambda I)$.

5. Let H be a Hilbert space and let $A: H \rightarrow H$ be a linear operator (matrix). If A is self-adjoint operator,

then all eigenvalues of A are real, i.e. $\sigma(A) \subset R$.

6. Spectral radius of a linear operator A on a finite-dimensional Hilbert space is:

$$r(A) = \max_{\lambda \in \sigma(A)} |\lambda|.$$

Example 1: Consider the linear operator represented by the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

2. Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

Since the matrix is triangular:

$$\det(A - \lambda I) = (2 - \lambda)^2(3 - \lambda)$$

2. Spectrum (eigenvalues):

$$(2 - \lambda)^2(3 - \lambda) = 0$$

Hence:

$$\sigma(A) = \{2, 3\}.$$

In this case, algebraic multiplicities are 2 and 1 for $\lambda = 2$ and $\lambda = 3$, respectively.

3. Geometric Multiplicity

As mentioned above, geometric multiplicity of A is $\dim \ker(A - \lambda I)$ for λ .

So, for $\lambda = 2$

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By solving the equation $(A - 2I)x = 0$, we will have $\ker(A - 2I) = \{x = (t, 0, 0) : t \in \mathbb{R}\}$ and $\dim \ker(A - \lambda I) = 1$. Consequently, geometric multiplicity of A is 1 for $\lambda = 2$.

For $\lambda = 3$

$$A - 3I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\ker(A - 3I) = \{x = (0, 0, t) : t \in \mathbb{R}\}$ and $\dim \ker(A - 3I) = 1$. Consequently, geometric multiplicity of A is 1 for $\lambda = 3$.

4. Spectral radius is

$$r(A) = \max\{|2|, |3|\} = 3.$$

Let us consider next examples for infinite case.

Example 2. Consider linear operator A on Hilbert space l_2 as:

$$A(x_1, x_2, x_3, \dots) = (2x_1, 3x_2, 4x_3, \dots).$$

By solving the equation $Ax = \lambda x$ we can get all eigenvalues as following: $\lambda = 2, 3, 4, \dots$ i.e. point spectrum consists of these numbers. We can show that continuous spectrum of this operator is empty. Indeed, for any $\lambda \in \mathbb{C} \setminus \{2, 3, 4, \dots\}$ the operator $A - \lambda I$ is invertible and R_λ is well defined on l_2 as well as bounded with

$$\begin{aligned} R_\lambda(y_1, y_2, y_3, \dots) &= (A - \lambda I)^{-1}(y_1, y_2, y_3, \dots) = \\ &= \left(\frac{1}{2-\lambda}y_1, \frac{1}{3-\lambda}y_2, \frac{1}{4-\lambda}y_3, \dots \right). \end{aligned}$$

So, set of regular points of A is $\mathbb{C} \setminus \{2, 3, 4, \dots\}$.

Example 3. Let us consider linear operator A on Hilbert space l_2 as:

$$A(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots).$$

By solving the equation $Ax = \lambda x$ we can get all eigenvalues as following: $\lambda = 1, \frac{1}{2}, \frac{1}{3}, \dots$ and these numbers belong to point spectrum, i.e. $\sigma_p(A) = \{\frac{1}{n} : n \in \mathbb{N}\}$. Continuous spectrum $\sigma_c(A)$ of A is $\sigma_c(A) = \{0\}$. Hence, the spectrum of A is

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) = \{\frac{1}{n} : n \in N\} \cup \{0\}.$$

Resolvent set $\rho(A)$ of A is $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

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INITIAL-NONLOCAL PROBLEM FOR A 2D WAVE EQUATION WITH A FRACTIONAL TIME DERIVATIVE

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Abstrakt. In this paper, we consider direct problems for a two-dimensional fractional wave equation with the Riemann-Liouville time fractional derivative. The direct problem is the initial-boundary problem for this equation with nonlocal boundary conditions. Using the method of separation of variables, a classical solution of direct problem was found in the form of a bi orthogonal series in terms of eigenfunctions and associated functions.

Keywords: two-dimensional fractional, wave equation, Riemann-Liouville fractional integral, inverse problem, spectral method, stability.

НАЧАЛЬНО-НЕЛОКАЛЬНАЯ ЗАДАЧА ДЛЯ ДВУМЕРНОГО ВОЛНОВОГО УРАВНЕНИЯ С ДРОБНОЙ ПРОИЗВОДНОЙ ПО ВРЕМЕНИ

Аннотация. В данной работе рассматривается прямая задача для двумерного волнового уравнения с дробной производной Римана–Лиувилля по времени. Прямая задача представляет собой начально-краевую задачу для данного уравнения с нелокальными граничными условиями. С использованием метода разделения переменных построено классическое решение прямой задачи в виде биортогонального ряда по собственным и присоединённым функциям.

Ключевые слова: двумерное дробное волновое уравнение, дробный интеграл Римана–Лиувилля, обратная задача, спектральный метод, устойчивость.

KASR VAQT HOSILASIGA EGA IKKI O'LCHAMLI TO'LQIN TENGLAMASI UCHUN BOSHLANG'ICH NOLOKAL-CHEGARAVIY MASALA

Annotatsiya. Ushbu ishda vaqt bo'yicha Riman–Liuvill ma'nosidagi kasr hosilaga ega ikki o'lchamli to'lqin tenglamasi uchun to'g'ri masala qaralgan. To'g'ri masala mazkur tenglama uchun nolokal chegaraviy shartli boshlang'ich-chegaraviy masaladan iborat. O'zgaruvchilarni ajratish usuli yordamida to'g'ri masalaning klassik yechimi xos funksiyalar va ularga mos qo'shma funksiyalar bo'yicha biortogonal qator ko'rinishida qurilgan.

Kalit so'zlar: ikki o'lchamli kasr tartibli to'lqin tenglamasi, Riman–Liuvill kasr integrali, teskari masala, spektral usul, turg'unlik.

Introduction. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major directions include anomalous diffusion, vibration and control, continuous time random walk, fractional Brownian motion, power law, fractional derivative and fractals, computational fractional derivative equations, nonlocal phenomena, biomedical engineering, fractional transforms, singularities analysis and integral representations for fractional differential systems, special functions related to fractional calculus, heat conduction, acoustic dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, fluid dynamics, and groundwater problems [1,2,3,4,5].

In the domain $\Omega := D \times (0, T]$, $D := \{(x, y): 0 < x, y < 1\}$ consider the time – fractional wave equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, y, t) - \Delta u(x, y, t) = f(x, y, t), (x, y, t) \in \Omega \quad (1)$$

with initial and boundary conditions

$$I_{0+\tau}^{(2-\alpha)} u(x, y, t) \Big|_{\tau=0} = \varphi(x, y), \quad (2)$$

$$\frac{\partial}{\partial t} \left(I_{0+t}^{(2-\alpha)} u \right) (x, y, t) \Big|_{t=0} = \psi(x, y), \quad (x, y) \in [0, 1] \times [0, 1], \quad (3)$$

$$u(0, y, t) = u(1, y, t), u_x(1, y, t) = 0, (y, t) \in [0, 1] \times [0, T], \quad (4)$$

$$u(x, 0, t) = u(x, 1, t) = 0, (x, t) \in [0, 1] \times [0, T]. \quad (5)$$

Here the Riemann-Liouville fractional differential operator $\frac{\partial^\alpha}{\partial t^\alpha}$ of the order $1 < \alpha < 2$ is defined as follows [[1], pp. 69-72, [4], pp. 62-65]

$$\frac{\partial^\alpha}{\partial t^\alpha} u(\cdot, \cdot, t) = \frac{\partial^2}{\partial t^2} \left(I_{0+t}^{(2-\alpha)} u \right) (\cdot, \cdot, t),$$

$$I_{0+t}^\gamma u(x, y, t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{u(x, y, \tau)}{(t-\tau)^{1-\gamma}} d\tau, \quad \gamma \in (0, 1),$$

is the Riemann-Liouville fractional integral of the function $u(x, y, t)$ with respect to t [1, pp. 69-72], $\Gamma(\cdot)$ is the Euler's Gamma function, Δ is the Laplace operator.

Everywhere in this paper, functions $f(x, y, t)$, $\varphi(x, y)$, $\psi(x, y)$ are known functions.

Assume that throughout this article, given functions φ, ψ, f, w and h satisfy the following conditions:

$$A_1) \{ \varphi, \psi \} \in C^2(\bar{D}), \left\{ \frac{\partial^3 \varphi(x, y)}{\partial x^i \partial y^j}, \frac{\partial^3 \psi(x, y)}{\partial x^i \partial y^j} \right\} \in L_2(D), i + j = 3; \varphi(0, y) = \varphi(1, y), \varphi_x(1, y) = 0,$$

$$\varphi_{xx}(0, y) = \varphi_{xx}(1, y), \varphi(x, 0) = \varphi(x, 1) = 0, \varphi_{yy}(x, 0) = \varphi_{yy}(x, 1) = 0;$$

$$\psi(0, y) = \psi(1, y), \psi_x(1, y) = 0, \psi_{xx}(1, y) = \psi_{xx}(1, y), \psi(x, 0) = \psi(x, 1) = 0, \psi_{yy}(x, 0) = \psi_{yy}(x, 1) = 0$$

$$A_2) f(x, y, \cdot) \in C_\sigma[0, T], t \in [0, T], f(\cdot, \cdot, t) \in C^2(\bar{D}), \frac{\partial^3 f(x, y, t)}{\partial x^i \partial y^j} \in L_2(D), i + j = 3; f(0, y, t) =$$

$$f(1, y, t), f_x(1, y, t) = 0, f_{xx}(0, y, t) = f_{xx}(1, y, t), f(x, 0, t) = f(x, 1, t) = 0, f_{yy}(x, 0, t) = f_{yy}(x, 1, t) = 0;$$

In section 2, we will provide some necessary preliminary informations.

Preliminaries. In this section we will give some definitions and results of differential calculus, which will be used repeatedly in the following sections.

In [8], the generalized Mittag-Leffler matrix function

$$E_{\alpha, \beta}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(\alpha k + \beta)}$$

was introduced, where $\alpha > 0, \beta \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers), and B is an arbitrary square matrix of order m . The generalized Mittag-Leffler matrix function plays an important role in studying fractional-order linear systems.

Proposition 1. Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \kappa \leq |\arg(z)| \leq \pi$$

For the proof, we refer to [[4].pp. 40-45], for example. We consider the weighted spaces of continuous functions [[1].pp. 4-5, 162-163].

$$C_\gamma[a, b] := \{g: (a, b) \rightarrow \mathbb{R}: (t-a)^\gamma g(t) \in C[a, b], 0 \leq \gamma < 1, \}$$

$$C_\gamma^\alpha(\Omega) = \left\{ g(t): \frac{\partial^\alpha}{\partial t^\alpha} g(t) \in C_\gamma(0, T]; 1 < \alpha \leq 2, \right\}$$

$$C_\gamma^{2, \alpha}(\Omega) = \{u(x, y, t): u(\cdot, \cdot, t) \in C^2(0, 1); t \in [0, T]$$

and

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, y, \cdot) \in C_\gamma(0, T]; x, y \in [0, 1], 1 < \alpha \leq 2, \}$$

$$C_\gamma^0[a, b] = C_\gamma[a, b]$$

with the norms

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_C, \|f\|_{C_\gamma^p} = \sum_{k=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_{C_\gamma}.$$

Theorem 1. (see [1, pp. 221-230]). The solution to the Cauchy type problem

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} u\right)(t) - \lambda u(t) = f(t), \left(\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} u\right)(a+) = b, \left(\frac{\partial^{\alpha-2}}{\partial t^{\alpha-2}} u\right)(a+) = d$$

with $1 < \alpha < 2$ and $\lambda, b, d \in \mathbb{R}$ has the form

$$u(t) = b(t-a)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-a)^\alpha] + d(t-a)^{\alpha-2} E_{\alpha,\alpha-1}[\lambda(t-a)^\alpha] + \int_a^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-\tau)^\alpha] f(\tau) d\tau,$$

In particular, the solution to the problem

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} u\right)(t) - \lambda u(t) = 0, \left(\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} u\right)(a+) = b, \left(\frac{\partial^{\alpha-2}}{\partial t^{\alpha-2}} u\right)(a+) = d,$$

is given by

$$u(t) = b(t-a)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-a)^\alpha] + d(t-a)^{\alpha-2} E_{\alpha,\alpha-1}[\lambda(t-a)^\alpha].$$

In the next section, we study the direct problem.

Investigation of the direct problem. Consider the following equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x, y, t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y, t) \in \Omega, \quad (6)$$

We will look for particular solutions to the problem (6), (4), (5) in the form

$$u(x, y, t) = \mathbb{Z}(x, y) v(t), \quad (7)$$

Substituting this expression into equation (6) and boundary conditions (4),(5) and separating the variables, we obtain the problem for finding the eigenfunctions

$$\frac{\partial^2 \mathbb{Z}}{\partial x^2} + \frac{\partial^2 \mathbb{Z}}{\partial y^2} + \mu^2 \mathbb{Z} = 0, \quad 0 < x, y < 1, \quad (8)$$

$$\mathbb{Z}(0, y) = \mathbb{Z}(1, y), \quad \frac{\partial \mathbb{Z}(1, y)}{\partial x} = 0, \quad (9)$$

$$\mathbb{Z}(x, 0) = \mathbb{Z}(x, 1) = 0, \quad 0 \leq x, y \leq 1, \quad (10)$$

here μ is the split parameter. Note that problem (8)-(10) is not self-adjoint in the sense of the scalar product $(\psi, \xi) = \iint_D \psi(x, y) \xi(x, y) dx dy$. It is easy to verify that the problem associated with it will be

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \mu^2 W = 0, \quad 0 < x, y < 1, \quad (11)$$

$$W(0, y) = 0, \quad \frac{\partial W(0, y)}{\partial x} = \frac{\partial W(1, y)}{\partial x}, \quad (12)$$

$$W(x, 0) = W(x, 1) = 0, \quad 0 \leq x, y \leq 1. \quad (13)$$

Let's solve problem (8)-(10). To do this, we represent its solution in the form

$$\mathbb{Z}(x, y) = X(x)Y(y). \quad (14)$$

Substituting this expression into equation (8) and boundary conditions (9),(10), we arrive at the problems

$$X''(x) + \lambda^2 X(x) = 0, \quad 0 < x < 1, \quad X(0) = X(1), X'(1) = 0, \quad (15)$$

$$Y''(y) + \gamma^2 Y(y) = 0, \quad 0 < y < 1, \quad Y(0) = Y(1) = 0, \quad (16)$$

where $\gamma = \mu - \lambda$. The solution to problem (16), as is known, has the form $\gamma_k = \pi k, Y_k(y) = \sqrt{2} \sin(\gamma_k y), k = 1, 2, \dots$. Here and below, the constants for eigenfunctions and associated functions are chosen from the normalization conditions.

The eigenvalues and the corresponding eigenfunctions of problem (15) were found in [6];

$$X_0 = 2, X_m(x) = 4 \cos(2\pi m x), \lambda_m = 2\pi m, m = 1, 2, \dots$$

Therefore, according to representation (14), the eigenvalues and eigenfunctions of problem (8)-(10) have the form $\mu_{mk}^2 = \lambda_m^2 + \gamma_k^2 = (2\pi m)^2 + (\pi k)^2, Z_{mk}(x, y) = X_m(x)Y_k(y), m = 0, 1, 2, \dots, k = 1, 2, \dots$. Note that the collection of eigenfunctions $Z_{mk}(x, y)$ is incomplete in the space $L_2(D)$, therefore, by analogy with [6], we complete the set of eigenfunctions with the associated functions $\tilde{Z}_{mk}(x, y), m = 0, 1, 2, \dots, k = 1, 2, \dots$, which are the solution to the problem

$$\frac{\partial^2 \tilde{Z}_{mk}}{\partial x^2} + \frac{\partial^2 \tilde{Z}_{mk}}{\partial y^2} + \mu_{mk}^2 \tilde{Z}_{mk} = p_m \mathbb{Z}_{mk}, \quad 0 < x, y < 1,$$

$$\tilde{Z}_{mk}(0, y) = \tilde{Z}_{mk}(1, y), \quad \partial \tilde{Z}_{mk}(1, y) / \partial x = 0, \quad 0 \leq y \leq 1,$$

$$\tilde{Z}_{mk}(x, 0) = \tilde{Z}_{mk}(x, 1) = 0, \quad 0 \leq x \leq 1,$$

where $p_m \neq 0$ is some constant. For $m = 0, k = 1, 2, \dots$, this problem has no solution. Setting $p_m = -2\lambda_m$, for $m, k = 1, 2, \dots$, we get

$$\tilde{Z}_{mk} = 4\sqrt{2}(1-x) \sin(2\pi mx) \sin(\pi ky),$$

Let us redesignate the system of eigenfunctions and associated functions of problem (8)-(10) as follows:

$$Z_{0k}(x, y) = X_0(x)Y_k(y), Z_{2m-1k}(x, y) = X_{2m-1}(x)Y_k(y), Z_{2mk}(x, y) = \tilde{Z}_{mk}. \quad (17)$$

The eigenfunctions and associated functions of the adjoint problem (11)-(13) have the form

$$W_{0k}(x, y) = xY_k(y), W_{2mk}(x, y) = \sin(2\pi mx)Y_k(y), \\ W_{2m-1k}(x, y) = x \cos(2\pi mx)Y_k(y), m, k = 1, 2, \dots \quad (18)$$

Note that the sequences of functions (17), (18) form a biorthonormal relationship: $(Z_{mk}, W_{lp}) = 1$ for $m = l, k = p$, otherwise $(Z_{mk}, W_{lp}) = 0$.

According to (14), particular solutions to problem (8)-(10) have the form

$$u(x, y, t) = \sum_{m,k=1}^{\infty} (Z_{mk}(x, y), U_{mk}(t)), \quad (19)$$

where $Z_{mk}(x, y) = (Z_{0k}(x, y), Z_{2mk}(x, y), Z_{2m-1k}(x, y))$, $U_{mk}(t) = (u_{0k}(t), u_{2mk}(t), u_{2m-1k}(t))$.

The coefficients of the vector $U_{mk}(t)$ are to be found by making use of the orthogonality of the eigenfunctions. Namely, we multiply (1) by the eigenfunctions of (18) and integrate over D . Recall that the scalar product in $L_2(D)$ is defined by

$$(f, g) = \iint_D f(x, y) g(x, y) dx dy.$$

Let us note the expansion coefficients of $f(x, y, t)$ and $\varphi(x, y), \psi(x, y)$ in the eigenfunctions of (18) for respectively by

$$(\varphi(x, y), W_{jk}(x, y)) = \varphi_{jk}, j = \{0, 2m, 2m-1\}, \quad (C1)$$

$$(\psi(x, y), W_{jk}(x, y)) = \psi_{jk}, j = \{0, 2m, 2m-1\}, \quad (C2)$$

$$(f(x, y, t), W_{jk}(x, y)) = f_{jk}(t), j = \{0, 2m, 2m-1\}. \quad (C3)$$

Taking into account (1)-(3) and C1)- C3), we obtain the following system of fractional-order differential equations for each fixed component $u_{0k}(t), u_{2mk}(t), u_{2m-1k}(t)$:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} U_{mk} + A_{mk} U_{mk}(t) = F_{mk}(t), \\ I_{0+,t}^{(2-\alpha)} U_{mk}(t) \Big|_{t=0} = \Phi_{mk}, \frac{d}{dt} (I_{0+,t}^{(2-\alpha)} U_{mk})(t) \Big|_{t=0} = \Psi_{mk}, \end{cases} \quad (20)$$

where

$$A_{mk} = \begin{pmatrix} \gamma_k^2 & 0 & 0 \\ 0 & \mu_{mk}^2 & 0 \\ 0 & 2\lambda_m & \mu_{mk}^2 \end{pmatrix}, \Phi_{mk} = \begin{pmatrix} \varphi_{0k} \\ \varphi_{2mk} \\ \varphi_{2m-1k} \end{pmatrix}, \Psi_{mk} = \begin{pmatrix} \psi_{0k} \\ \psi_{2mk} \\ \psi_{2m-1k} \end{pmatrix}, F_{mk}(t) = \begin{pmatrix} f_{0k}(t) \\ f_{2mk}(t) \\ f_{2m-1k}(t) \end{pmatrix}.$$

Based on Theorem 1, we have that the initial problems (20) is equivalent in the space $C_\sigma^\alpha[0, T]$ to the Volterra integral equation of the second kind

$$U_{mk}(t) = t^{\alpha-2} E_{\alpha, \alpha-1}(-A_{mk} t^\alpha) \Phi_{mk} + t^{\alpha-1} E_{\alpha, \alpha}(-A_{mk} t^\alpha) \Psi_{mk} + \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-A_{mk}(t-\tau)^\alpha) F_{mk}(\tau) d\tau, \quad (21)$$

Where

$$E_{\alpha, \alpha}(-A_{mk} t^\alpha) = \mathbb{T} E_{\alpha, \alpha}(-J t^\alpha) \mathbb{T}^{-1} = \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2\lambda_m} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_{\alpha, \alpha}(-\gamma_k^2 t^\alpha) & 0 & 0 \\ 0 & E_{\alpha, \alpha}(-\mu_{mk}^2 t^\alpha) & \frac{d}{dz} E_{\alpha, \alpha}(z) \Big|_{z=-\mu_{mk}^2} t^\alpha \\ 0 & 0 & E_{\alpha, \alpha}(-\mu_{mk}^2 t^\alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2\lambda_m & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} E_{\alpha,\alpha}(-\gamma_k^2 t^\alpha) & 0 & 0 \\ 0 & E_{\alpha,\alpha}(-\mu_{mk}^2 t^\alpha) & 0 \\ 0 & 2\lambda_m \frac{d}{dz} E_{\alpha,\alpha}(z) \Big|_{z=-\mu_{mk}^2 t^\alpha} & E_{\alpha,\alpha}(-\mu_{mk}^2 t^\alpha) \end{pmatrix}.$$

Lemma 1. We have the estimates:

$$\|U_{mk}\|_{C_\sigma[0,t]} \leq C\lambda_m(t^{\sigma+\alpha-2}|\Phi_{mk}| + t^{\sigma+\alpha-1}|\Psi_{mk}| + \|F_{mk}\|_{C_\sigma[0,T]}t^\alpha) = \mathbb{V}_{mk}(t);$$

$$t^\sigma \left| \frac{\partial^\alpha}{\partial t^\alpha} U_{mk} \right| \leq \|F_{mk}\|_{C_\sigma[0,t]} + \mu_{mk}^2 \mathbb{V}_{mk}(t),$$

where $2 - \alpha < \sigma < 1$, $\|U_{mk}\|_{C_\sigma[0,t]} = \max_{0 \leq \tau \leq t} \{|\tau^\sigma U_{mk}(\tau)|\}$, $B(\alpha, 1 - \sigma)$ is Euler's beta function and

$$\|\Phi_{mk}\| = \max\{\max\{|\varphi_{0k}|\}, \max\{|\varphi_{2mk}|\}, \max\{|\varphi_{2m-1k}|\}\},$$

$$\|\Psi_{mk}\| = \max\{\max\{|\psi_{0k}|\}, \max\{|\psi_{2mk}|\}, \max\{|\psi_{2m-1k}|\}\},$$

$$\|F_{mk}\|_{C_\sigma[0,t]} = \max\left\{\max_{0 \leq \tau \leq t} \{|\tau^\sigma f_{0k}(\tau)|\}, \max_{0 \leq \tau \leq t} \{|\tau^\sigma f_{2mk}(\tau)|\}, \max_{0 \leq \tau \leq t} \{|\tau^\sigma f_{2m-1k}(\tau)|\}\right\},$$

Proof. The solution of (21) is bounded in $C_\sigma^\alpha[0, T]$ in view of A1), A2). Multiplying the last equation (21) by t^σ , we get

$$\begin{aligned} \|U_{mk}\|_{C_\sigma[0,t]} &\leq t^{\sigma+\alpha-2} |E_{\alpha,\alpha-1}(-J_{mk}t^\alpha)\Phi_{mk}| + t^{\sigma+\alpha-1} |E_{\alpha,\alpha-1}(-J_{mk}t^\alpha)\Psi_{mk}| + \\ &t^\sigma \int_0^t (t-\tau)^{\alpha-1} |E_{\alpha,\alpha}(-J_{mk}(t-\tau)^\alpha)F_{mk}(\tau)| d\tau \leq \\ &\leq C\lambda_m t^{\sigma+\alpha-2} \|\Phi_{mk}\| + C\lambda_m t^{\sigma+\alpha-1} \|\Psi_{mk}\| + C\lambda_m \|F_{mk}\|_{C_\sigma[0,t]} t^\alpha. \end{aligned}$$

Next, according to Lemma 1, we get

$$\|U_{mk}\|_{C_\sigma[0,t]} \leq C\lambda_m(t^{\sigma+\alpha-2}\|\Phi_{mk}\| + t^{\sigma+\alpha-1}\|\Psi_{mk}\| + \|F_{mk}\|_{C_\sigma[0,t]}t^\alpha).$$

The second part of Lemma 1 is obtained from the equation of problem (20). We will look for a solution to the problem (1)-(5) in the form of (19) double Fourier series. We compute $t^\sigma u(x, y, t)$, $t^\sigma \left(\frac{\partial^\alpha}{\partial t^\alpha} u\right)(x, y, t)$, $t^\sigma u_{xx}(x, y, t)$, $t^\sigma u_{yy}(x, y, t)$ and estimates. As result we obtain

$$|t^\sigma u(x, y, t)| \leq C \sum_{m,k=1}^{\infty} \lambda_m (T^{\sigma+\alpha-2} \|\Phi_{mk}\| + T^{\sigma+\alpha-1} \|\Psi_{mk}\| + \|F_{mk}\|_{C_\sigma[0,T]} T^\alpha); \quad (22)$$

$$\left| t^\sigma \frac{\partial^\alpha}{\partial t^\alpha} u(x, y, t) \right| \leq \sum_{m,k=1}^{\infty} [\|F_{mk}\|_{C_\sigma[0,T]} + \mu_{mk}^2 \tilde{V}_{mk}(T)]; \quad (23)$$

$$|t^\sigma u_{xx}(x, y, t)| \leq C \sum_{m,k=1}^{\infty} \lambda_m^3 (T^{\sigma+\alpha-2} \|\Phi_{mk}\| + T^{\sigma+\alpha-1} \|\Psi_{mk}\| + \|F_{mk}\|_{C_\sigma[0,T]} T^\alpha); \quad (24)$$

$$|t^\sigma u_{yy}(x, y, t)| \leq C \sum_{m,k=1}^{\infty} \gamma_k^2 \lambda_m (T^{\sigma+\alpha-2} \|\Phi_{mk}\| + T^{\sigma+\alpha-1} \|\Psi_{mk}\| + \|F_{mk}\|_{C_\sigma[0,T]} T^\alpha); \quad (25)$$

In view of Lemma 1, if the following series converges, then the series (22)-(25) will converge for any $(x, y, t) \in \bar{\Omega}$:

$$C_4 \sum_{m,k=1}^{\infty} \mu_{mk}^3 (\|F_{mk}\|_{C_\sigma[0,T]} + \|\Phi_{mk}\| + \|\Psi_{mk}\|), \quad (26)$$

where the constant C_4 depends only on T, α, σ . We now state the following auxiliary lemma

Lemma 2. If conditions $A_1), A_2)$ are satisfied, then the numerical series in (26) is converge.

Theorem 2. Let $q(t) \in C[0, T], A_1), A_2)$ hold, then there exists a unique solution to the direct problem (1)-(5) $u(x, y, t) \in C_{xy,t}^{2,\alpha}(\bar{\Omega})$.

Conclusion. In this paper, we study the solvability of a direct problem for a two-dimensional wave equation with a fractional Riemann-Liouville time derivative with initial non-local boundary conditions. The local existence and global uniqueness of the solution to the stability of the direct problem is proved. Non-

local boundary conditions, the fractional Riemann-Liouville derivative, and the control coefficient complicated our task.

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